Course Contents:

Chapter 1: *Sets, sequences and series:* Ordered set, Existence theorem, Archimedean property, Density property, Extended real number system, Euclidean Spaces, Countable and Uncountable sets, Cantor sets, sequence and series of numbers: infimum, supremum, limits, convergence and divergence of series of numbers, Root and Ratio test, power series, radius of convergence, absolute convergence

Chapter 2: *Basic Topology:* Metric spaces, Open, closed, compact, perfect, connected sets

Chapter 3: *Continuity:* limits of functions, continuous functions, continuity and compactness, continuity and connectedness, discontinuity, monotonic functions, limits at infinity, bounded variations

Chapter 4: *Differentiation:* derivative of real functions, Mean value theorems, L'Hopital's rule, Continuity of derivative, Taylor's theorem, rectifiable curves

Chapter 5: *Integration:*Definition and existence of integral, properties of integrals, integration and differentiation (fundamental theorem, integration by parts)

Chapter 6: *Sequence and series of functions:* examples, uniform convergence, and continuity, and integration, and differentiation, equicontinuity, Arzela-Ascoli theorem, Stone Weierstrass theorem

Chapter 7: *(Wont be taught in detail:)Special functions:* Power series, Taylor's theorem, Exponential, Lograthmic, trigonometric functions, Gamma function, Fourier series

Notations

$\mathbb{N}$: Set of all natural numbers
$\mathbb{Z}$: Set of all integers
$\mathbb{Q}$: Set of all rational numbers
$\mathbb{R}$: Set of all real numbers
$\mathbb{R}^*$: Extended real numbers $(\mathbb{R} \cup \{\infty, -\infty\})$

1. Sets, sequences and series

1.1. Sets and Numbers.

Definition 1.1. *Let S be a set. An order on S is a relation, denoted by $<$., with the following two properties:*

(i) *If $x \in S$ and $y \in S$ then one and only one of the statements*

$$x < y \quad x = y, \quad y < x$$

is true.

(ii) *If $x, y, z \in S$ such that $x < y$ and $y < z$, then $x < z$.*

Definition 1.2. *An ordered set is a set S in which an order is defined.*

For example, $\mathbb{Q}$ is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive number.

Definition 1.3. *Suppose S is an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is* bounded above *and call β an upper bound of E.*
Lower bounds are defined in the same way.

Definition 1.4. *Suppose S is an ordered set, $E \subset S$ and E is bounded above and there exists an $\alpha \in S$ with the following properties:*

(i) *α is an upper bound of E.*

(ii) *If $\gamma < \alpha$ then γ is not an upper bound of E.*

Then α is called the least upper bound *of E or the* supremeum *of E, and we write $\alpha = \sup E$. Similarly, the* greatest lower bound *or* infimum *of a set E which is bounded below is defined.*

Example: Let E consist of all numbers $1/n$ where $n \in \mathbb{N}$. Then $\sup E = 1$ which is in E and $\inf E = 0$, which is not in E.

Definition 1.5. *An ordered set S is said to have the* least upper bound property *if the following is true:*
If $E \subset S$ is non-empty and bounded above then $\sup E$ exists in S.

We call a set "*field*" if it satisfies a set of axioms. (We will see in detail about field in Algebra). An ordered field is a field S which is also an ordered set such that (i) $x + y < x + z$ if $x, y, z \in S$ and $y < z$, (ii) $xy > 0$ if $x, y \in S$, $x > 0$ and $y > 0$. $\mathbb{Q}$ is an ordered field.

Theorem 1.6. ***Existence theorem:*** *There exists an ordered field R which has the least upper bound property. Moreover R contains Q as a subfield. (We call this ordered field $\mathbb{R}$-The real numbers)*

Proof: self-read - Page 17-21 in [**?**]

Theorem 1.7. *Archimedean property of $\mathbb{R}$: If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$.*

Proof. Suppose the theorem is false. Then y is an upper bound of the set $S = \{nx : n \in \mathbb{N}\}$. By the Existence theorem, S has a least upper bound in $\mathbb{R}$. Put $\alpha = \sup S$. Since $x > 0$, we have

$$\alpha - x < \alpha.$$

By the definition of α, $\alpha - x$ is not an upper bound of S. Hence there exists $m \in \mathbb{N}$ such that

$$\alpha - x < mx.$$

That is, $\alpha < (1+m)x \in S$. This contradicts the definition of α. Hence the theorem. □

Theorem 1.8. *Density property: If $x, y \in \mathbb{R}$ such that $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$.*

Proof. Since $x < y$, $y - x > 0$ and by archimedean property there exists positive integers n, m_1, m_2 such that

$$\begin{aligned} n(y-x) &> 1, \text{ that is, } ny > 1 + nx \\ m_1 &> nx \\ m_2 &> -nx. \end{aligned} \tag{1}$$

Hence we have

$$-m_2 < nx < m_1.$$

Hence there is an integer m (with $-m_2 \le m \le m_1$) such that $m - 1 \le nx < m$. Thus

$$nx < m \le 1 + nx < ny.$$

Since $n > 0$, it follows that

$$x < \frac{m}{n} < y.$$

With $p = \frac{m}{n}$, hence the theorem. □

Theorem 1.9. *Existence of nth roots of positive reals: For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.*

This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

Proof. **Existence:** Let E be the set consisting of all positive real numbers t such that $t^n < x$, that is, $E = \{t : t \in \mathbb{R},\ t > 0,\ t^n < x\}$. E is non-empty, since, consider $t = \frac{x}{1+x}$: $0 \le t < 1$ and hence $t^n < t < x$, that is, $t \in E$. If $s > 1 + x$ then $s^n \ge s > x$, so that $s \notin E$. Thus $1 + x$ is an upper bound of E. By Existence theorem, there exists $y = \sup E$.

To prove that $y^n = x$, we will show that each of the inequalities $y^n < x$ and $y^n > x$ leads to contradiction.

We have $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + ... + a^{n-1})$. If $a < b$, then $b^n - a^n < (b-a)bn^{n-1}$.

Assume $y^n < x$**.** That is, $x - y^n > 0$ and hence $\frac{x-y^n}{n(y+1)^{n-1}} > 0$. Choose $0 < h < \frac{x-y^n}{n(y+1)^{n-1}}$. Then

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$

Thus $(y+h)^n < x$, and $y + h \in E$. Since $y + h > y$, this contradicts the fact that y is an upper bound of E.

Assume $y^n > x$. Then $\frac{y^n-x}{ny^{n-1}} > 0$ and also $(n-1)y^n > x$, that is, $(1-n)y^n < -x$ and hence $k = \frac{y^n-x}{ny^{n-1}} < y$. We have

$$y^n - (y-k)^n < kny^{n-1} = y^n - x.$$

Thus $x < (y-k)^n$. That is $y-k$ is an upper bound of E. But $y-k < y$. This contradicts the fact that y is the least upper bound of E.

Hence $y^n = x$.

Uniqueness: Suppose there exists more that one real number, say $0 < y_1 < y_2$ such that $y_1^n = x = y_2^n$. Since $0 < y_1 < y_2$, $y_1^n < y_2^n$, that is $x < x$ which is not true. Hence there exists at most one such y.

Hence the proof. □

Euclidean Spaces:

Definition 1.10. *For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered $k-$tuples*

$$\boldsymbol{x} = (x_1, x_2, ..., x_k),$$

where $x_1, x_2, ...x_k$ are real numbers, called the coordinates of $\boldsymbol{x}$. The elements of $\mathbb{R}^k$ are called points, or vectors (when $k > 1$). If $\boldsymbol{y} = (y_1, y_2, ...y_k)$ and if c is a real number,

$$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &= (x_1 + y_1, ...x_k + y_k), \\ c\boldsymbol{x} &= (cx_1, ...cx_k) \end{aligned}$$

so that $\boldsymbol{x} + \boldsymbol{y} \in \mathbb{R}^k$ and $c\boldsymbol{x} \in \mathbb{R}^k$.
Inner product: $\boldsymbol{x}.\boldsymbol{y} = \sum_{i=1}^k x_i y_i$
Norm of $\boldsymbol{x}$: $|\boldsymbol{x}| = (\boldsymbol{x}.\boldsymbol{x})^{1/2} = \left(\sum_i^k x_i^2\right)^{1/2}$.
$\mathbb{R}^k$ together with the inner product and norm defined is called Euclidean $k-$space.

Recall: f is said to be a function from a set A to a set B if for every element $a \in A$ there is associated in some manner, an element $b \in B$ denoted by $f(a)$. A-Domain of f and $f(A)$-set of all values of f is called the range of f or image of A under f.

If $f(A) = B$ then f maps A onto B.
$f^{-1}(E)$ for $E \subset B$ is $\{x \in A : f(x) \in E\}$ called the inverse image of E under f.
If $a_1, a_2 \in A$ with $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ then f is called one-one mapping of A into B.

Definition 1.11. *If there exists a one-one onto mapping from A to B then we say that A and B have the same cardinal numbers or A and B are equivalent. We write $A \sim B$.*

For any positive integer n, $J_n = \{1, 2...n\}$.

1. A is finite if A J_n for some n
2. A is infinite if A is not finite.
3. A is countable or enumerable or denumerable if A $\mathbb{N}$.
4. A is uncountable if A is neither finite nor countable.
5. A is at most countable if A is finite or countable.

Example: $\mathbb{Z}$ $\mathbb{N}$: Consider the following function from $\mathbb{N}$ to $\mathbb{Z}$: $f(n)$ is defined as

$$\frac{n}{2} \text{ (if } n \text{ is even and) } \frac{-(n-1)}{2} \text{(if } n \text{ is odd.)}$$

Theorem 1.12. *Every infinite subset of a countable set A is countable.*

Proof. Suppose $E \subset A$ and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:
Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, ..., n_{k-1}$ $(k = 2, 3..)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$. Putting $f(k) = x_{n_k}$, $(k = 1, 2...)$ we obtain a $1-1$ correspondence between E and J. Hence the proof. □

Theorem 1.13. *Countable union of countable sets is countable*

Proof. Let $\{E_n\}_{n=1,2...}$ ne a sequence of countable sets and let $S = \cup_{n=1}^{\infty} E_n$. We claim that S is countable.
Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2...$ and consider the infinite array

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

in which the elements of E_n form the n^{th} row. These elements can be arranged in a sequence

$$x_{11}; x_{12}, x_{21}; x_{13}, x_{22}, x_{31}; x_{14}, x_{23}, x_{32}, x_{41}; ....$$

If any two of the sets E_n have elements in common, these will appear more than once in the above sequence. Hence there is a subset T of the set of all positive integers such that S T which shows that S is at most countable (by the theorem **??**]. Since $E_1 \subset S$ and E_1 is infinite, S is infinite and thus countable. □

The Cantor Set: We construct the Cantor set in the following way:
Let E_0 be the interval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$ and let E_1 be the union of the intervals $[0, \frac{1}{3}]$ $[\frac{2}{3}, 1]$. Remove the middle thirds of these intervals and let E_2 be the union of the intervals

$$[0, \frac{1}{9}],\ [\frac{2}{9}, \frac{3}{9}], [\frac{6}{9}, \frac{7}{9}],\ [\frac{8}{9}, 1].$$

Continuing this way, we obtain a sequence of sets E_n such that

(a) $E_1 \supset E_2 \supset E_3 \supset ...$
(b) E_n is the union of 2^n intervals each of length 3^{-n}.

The set $P = \cap_{n=1}^{\infty}$ is called the **Cantor set**.
Q: Is P non-empty? (Yes, we will justify in the next chapter.)

1.2. **Sequence and series of numbers.**

Recall:

(1) A sequence $\{p_n\}$ is said to converge if there is a point $p \in \mathbb{R}$ with the following property: For every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies that $|p_n - p| < \epsilon$. We call p the limit of the sequence $\{p_n\}$, $\{p_n\}$ converges to p. Notation: $lim_{n\to\infty} p_n = p$.
(2) If $\{p_n\}$ does not converge, then it is said to diverge.
(3) The set of all points p_n, $n = 1, 2..$ is called the range of $\{p_n\}$.
(4) If the range of a sequence is bounded then we call the sequence a bounded sequence.
(5) A sequence $\{p_n\}$ in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $|p_n - p_m| < \epsilon$ if $n \geq N$ and $m \geq N$.
(6) Given a sequence $\{a_n\}$, we denote $s_n = \sum_{k=1}^{n} a_k$ as the partial sums of the series $\sum_{n=1}^{\infty} a_n$. If the sequence $\{s_n\}$ of the partial sums of the series converges to s then we write $\sum_{n=1}^{\infty} s_n = s$ and s is called the limit of a sequence of sums. If $\{s_n\}$ diverges then the series diverges.
(7) The number e: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$, where $n! = 1.2.3...n$ for $n \geq 1$ and $0! = 1$. (Since their partial sums $s_n < 3$, that is, bounded and monotonic, we have $\{s_n\}$ converges. By uniqueness of limit, we have e as well defined number.)
(8) Given a sequence $\{c_n\}$ of complex numbers the series $\sum_{n=0}^{\infty} c_n z^n$ is called a power series. The numbers c_n are called the coefficients of the series.
(9) The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Theorem 1.14. *Cauchy Criterion:*

(1) *Every convergent sequence is a Cauchy sequence.*
(2) *In $\mathbb{R}$, every Cauchy sequence converges. (Same holds for $\mathbb{R}^k$)*

Proof. (1) Suppose $p_n \to p$. That is, if $\epsilon > 0$ there there is an integer N such that $|p_n - p| < \epsilon/2$ for all $n \geq N$. Hence

$$|p_n - p_m| = |p_n - p + p - p_m| \leq |p_n - p| + |p_m - p| \leq \epsilon$$

whenever $n \geq N$ and $m \geq N$. Hence $\{p_n\}$ is Cauchy.

(2) We will see in in the next chapter.

□

Theorem 1.15. *(Uniqueness of limit:) If p and q are two numbers such that $\{p_n\}$ converges to p and q, then $p = q$.*

Proof. Let $\epsilon > 0$ be given. Then there exists integers N and N' such that $n \geq N$ implies $|p_n - p| < \epsilon/2$ and $n \geq N'$ implies $|p_n - q| < \epsilon/2$. Hence if $n \geq max\{N, N'\}$ then

$$|p - q| = |p - p_n + p_n - q| \leq |p_n - p| + |p_n - q| < \epsilon.$$

□

Theorem 1.16. *If $\{p_n\}$ converges then $\{p_n\}$ is bounded.*

Proof. Suppose $p_n \to p$ then there exists an integer N such that for all $n \geq N$, $|p_n - p| < 1$. Put $r = max\{1, |p_1 - p|, |p_2 - p|, ...|p_N, p|\}$. Then $|p_n - p| \leq r$ for all n. This $|p_n| = |p_n - p + p| \leq |p_n - p| + p \leq r + p$. $r + p$ is the upper bound. □

Theorem 1.17. *Let $\{s_n\}$ be a monotonic sequence, that is either for all n, $s_n \leq s_{n+1}$ monotonically increasing or for all n, $s_n \geq s_{n+1}$ monotonically decreasing. Then $\{s_n\}$ converges if and only if it is bounded.*

Proof. Without loss of generality, we assume that $\{s_n\}$ is monotonically increasing. Suppose $\{s_n\}$ is bounded. Let s be the least upper bound of the set of all points s_n. Then $s_n \leq s$ for all n. For every $\epsilon > 0$ we have $s - \epsilon < s$ and since s is the least upper bound, there exists N such that $s - \epsilon < s_N \leq s$. Since $\{s_n\}$ increases, for all $n \geq N$ we have $s - \epsilon < s_n \leq s$. Thus $\{s_n\}$ converges to s. Conversely, suppose $\{s_n\}$ converges to s. We just saw that it is bounded. □

Homework/Self read: Algebraic operations of sequences: What about sum/difference/product/division of convergent sequences,

Theorem 1.18. *If $\sum a_n$ converges, then $lim_{n\to\infty} a_n = 0$.*

The converse of the above theorem is not true. Consider $\sum_1^\infty \frac{1}{n}$.

Proof. Enough to prove: (Since put $m = n$ in the following statement) $\sum a_n$ *converges if and only if for every $\epsilon > 0$ there is an integer N such that*

$$|\sum_{k=n}^{m} a_k| \leq \epsilon$$

if $m \geq n \geq N$. This statement is evident from Cauchy criterion. □

Theorem 1.19. *e is irrational.*

Proof. Suppose e is rational, that is $e = p/q$ where p and q are positive integers. Consider the partial sum $s_n = \sum_{k=0}^{n}$.

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + ... \\ &< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + ...\right) = \frac{1}{n!n} \end{aligned}$$

That is $0 < (e - s_q) < \frac{1}{q!q}$. Hence

$$0 < q!(e - s_q) < \frac{1}{q}. \tag{2}$$

By our assumption, $q!e$ is an integer. Also

$$q!s_q = q!(1 + 1 + \frac{1}{2!} + ...\frac{1}{q!})$$

is an integer. Thus $q!(e - s_q)$ is an integer. Since $q \geq 1$, (**??**) implies that there is an integer between 0 and 1 which is a contradiction. □

Theorem 1.20. *Root Test: Given $\sum a_n$. Let $\alpha = \underset{n\to\infty}{limsup} \sqrt[n]{|a_n|}$. Then*

(1) *if $\alpha < 1$ then $\sum a_n$ converges;*
(2) *if $\alpha > 1$ then $\sum a_n$ diverges;*
(3) *if $\alpha = 1$ the test gives no information.*

Proof. (1) Suppose $\alpha < 1$. By the property of $limsup$ we can choose β such that $\alpha < \beta < 1$ and an integer N such that $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$. That is for $n \geq N$, $|a_n < \beta^n|$. Since $0 < \beta < 1$, $\sum \beta^n$ converges. By comparison test, $\sum a_n$ converges.

(2) Suppose $\alpha > 1$. Again by property of $limsup$ there is a sequence $\{n_k\}$ such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. Hence $|a_n| > 1$ for infinitely many values of n, so that the condition $a_n \to 0$, (the condition which is necessary for convergence of $\sum a_n$,) does not hold. Hence the proof by Theorem **??**.

(3) Consider $\sum \frac{1}{n}$. The series diverges and $\alpha = 1$. Also consider the series $\sum \frac{1}{n^2}$. It converges and $\alpha = 1$.

□

Theorem 1.21. *Given the power series $\sum c_n z^n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and $R = \frac{1}{\alpha}$. Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$. (R is called the radius of convergence.)*

Proof. Apply Root test with $a_n = c_n z^n$. □

Q? When can you speak about the power series convergence when the radius of convergence? (THeorem 3.44 in Rudin)

Theorem 1.22 (**If time permits...discuss:**). *Summation by parts: Given two sequences $\{a_n\}$, $\{b_n\}$. Put $A_n \sum_{k=0}^{n} a_k$ if $n \geq 0$ and $A_{-1} = 0$. Then $0 \leq p \leq q$, we have*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p. \tag{3}$$

Theorem 1.23 (**If time permits discuss else during extra class**). *Suppose*

(1) *the partial sums A_n of $\sum a_n$ form a bounded sequence;*
(2) $b_0 \geq b_1 \geq b_2 \geq ...$;
(3) $lim_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

Proof. Choose M such that $|A_n| \leq M$ for all n. Given $\epsilon > 0$, there is an integer N such that $b_N \leq (\epsilon/2M)$. For $N \leq p \leq q$, we have

$$\begin{aligned} \left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq M\left|\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p\right| \\ &= 1M b_p \leq 2M b_N \leq \epsilon. \end{aligned}$$

Convergence now follows from the Cauchy criterion. We note that the first inequality in the above chine depends on the fact that $b_n - b_{n+1} \geq 0$. □

Theorem 1.24. *Absolute convergence implies convergence. (*Proof follows from Cauchy criterion.*)*

The converse of the above theorem is not true. For example $\sum \frac{(-1)^n}{n}$ converges absolutely but diverges.

2. Basic Topology

What have we studied so far? We have constructed the set of real numbers (by Archimedian property and density property). With that then we moved on to what is sequence and series. Instead of considering sequences and series of real numbers, we mentioned we will generalize the fact to 'metric spaces'. We will learn about it..

Definition 2.1. *A set X whose elements we call points is said to metric space if with any two points p and q of X there is associated a real number $d(p,q)$ called the distance from p to q such that*

1. $d(p,q) > 0$ *if* $p \neq q$; $d(p,p) = 0$
2. $d(p,q) = d(q,p)$
3. $d(p,q) \leq d(p,r) + d(r,q)$ *for any* $r \in X$.

Any function with these three properties is called a distance function, or a metric.

For example, for $x = (x_1, ...x_k)$ and $y = (y_1, ...y_k)$ in $\mathbb{R}^k$, $d(x,y) = |x - y|$.

Let X be a metric space.

1. A neighborhood of p is a set $N_r(p)$ consisting of all q such that $d(p,q) < r$ for some $r > 0$. The number r is called the radius of $N_r(p)$. *Let us think of* $(p-r, p+r)$ *for any* $p \in \mathbb{R}$
2. A point p is a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$. *In* $\mathbb{R}$ *for any* $r > 0$, $(p-r, p+r) \cap E$ *has a point other than* p.
3. If $p \in E$ and p is not a limit point of E then p is called an isolated point.
4. E is closed if every limit point of E is a point of E.*Note that a finite set in* $\mathbb{R}$ *has no limit point in* $\mathbb{R}$ *and hence closed.*
5. A point p is an interior point of E if there is a neighborhood N of p such that $N \subset E$.*A point* p *is an interior point of a set* E *in* $\mathbb{R}$ *if and only if we can find an interval in that set* E *containing that point* p.
6. E is open if every point of E is an interior point of E.
7. The complement of E denoted by E^c is the set of all points $p \in X$ such that $p \notin E$.
8. E is perfect if E is closed and if every point of E is a limit point of E.
9. E is bounded if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.
10. E' denotes the set of limit points of E in X. The *closure* of E is $\overline{E} = E \cup E'$.
11. E is dense in X if every point of X is a limit point of E or a point of E (or both).
12. Two subsets A and B of a metric space X are said to be **separated** if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty. (that is if no point of A lies in the closure of B and no point of B lies in the closure of A.) A set $E \subset X$ is said to be connected if E is not a union of two nonempty separated sets.

Theorem 2.2. *(Just hint the proof:) If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Proof. Suppose there is a neighborhood N of p which contains only finite points $\{q_1, ...q_N\}$ such that $q_j \neq p$, then for $r = \min_{1 \leq j \leq N} d(p, q_j) > 0$, the neighborhood $N_r(p)$ contains no point other than p which contradicts the definition of limit point. Hence the proof. □

Theorem 2.3. *If $A \subset \mathbb{R}$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in A such that $p = lim_{n\to\infty} p_n$.*

Proof. By the definition, there is a point $p_n \in E$ such that $|p_n - p| < 1/n$. Given $\epsilon > 0$ choose N such that $N\epsilon > 1$. For all $n > N$ then we have $|p_n - p| < \epsilon$. Hence the proof. □

Theorem 2.4. *A set E is open if and only if its complement E^c is closed.*

Proof. Suppose E^c is closed. To prove E is open, we have to prove that any point in E is an interior point of E, that is, if $x \in E$ then we have to prove that x is an interior point of E.
Since $x \in E$, we have $x \notin E^c$, that is, x is not a limit point of E^c. Thus there exists a neighborhood N of x that contains no point in E^c, that is $N \subset E$. Thus x is an interior point of E.
Now suppose E is open. Then to prove E^c is closed, we have to prove that every limit point of E^c is a point of E. Let x be a limit point of E^c, that is, every neighborhood of x intersects with E^c other than x, which means every neighborhood of x is not contained in E. Thus x is not an interior point of E. Hence $x \notin E$, that is, $x \in E^c$. Hence the proof. □

Corollary 2.5. *A set F is closed if and only if its complement is open.*

Theorem 2.6. *(Just State and give hints to proof:) If X is a metric space and $E \subset X$, then*

(a) *$\overline{E}$ is closed*
(b) *$E = \overline{E}$ if and only if E is closed.*
(c) *$\overline{E}$ is the smallest closed set that contains E.*

Proof.

(a) Enough to prove $\overline{E}^c$ is open. Let $x \notin \overline{E}^c$. Then $x \notin E$ and x is not a limit point of E. Hence we can find a neighborhood N of x that never intersects $\overline{E}$, that is $N \subset E$. Hence x is an interior point of E.
(b) $E = \overline{E}$ implies that every limit point of E is in E and hence E is closed. Suppose E is closed. We already have $E \subset \overline{E}$. Enough to prove that if x is a limit point of E then $x \in E$ which is true since E is closed.
(c) Suppose $(E \subset)F$ is a closed set. Clearly $E \subset \overline{E}$. To prove: $\overline{E} \subset F$. $E \subset F$ implies $E' \subset F'$ and since F is a closed set, $E' \subset F' \subset \overline{F} = F$. Thus $E \subset F$ and $E' \subset F$ implies the theorem.

□

Theorem 2.7. *Let E be a nonempty set of real numbers which is bounded above. Let $y = sup\ E$. Then $y \in \overline{E}$. Hence, $y \in E$ if E is closed.*

Proof. If $y \in E$ then there is nothing to prove. Suppose $y \notin E$. By the definition of *sup*, for every $\epsilon > 0$ there exists a point $x \in E$ such that $y - h < x < y$, that is, the neighborhood $(y - h, y + h)$ intersects E other than y. Hence y is a limit point of E, that is $y \in E'$. Hence $y \in \overline{E}$. □

Theorem 2.8. *Homework: The set of all subsequential limit points of a sequence x_n in a metric space X is a closed subset of X.*

Compact sets:

(1) An open cover of a set E in a metric space X is a collection $\{G_\alpha\}$ of arbitrary open subsets of X such that $E \subset \cup_\alpha G_\alpha$.
(2) A set K in a metric space X is said to be compact if every open cover of K is a finite subcover.

Theorem 2.9. *Compact sets of metric space X is closed.*

Proof. Let K be a compact set. To prove that K is closed. We prove that K^c is open, that is to prove that every point p in K^c is an interior point of K^c.
Let $q \in K$. With radius less than $\frac{1}{2}d(p,q)$, V_q and W_q be the neighborhood of p and q respectively. Then $K \subset \cup_{q\in K} W_q$. Since K is compact, there exists $q_1, ...q_n$ such that $K \subset \cup_{j=1}^n W_{q_j}$. Also $p \in V = \cap_{j=1}^n V_{q_j}$ does not intersect with $\cup_j W_{q_j}$ and hence does not intersect with K, that is V is a neighborhood of p contained in K^c. Thus p is an interior point of K^c. □

Theorem 2.10. *(state and hint:) Closed subsets of compact sets is compact. In particular, if F is closed and K is compact, then $F \cap K$ is compact.*

Theorem 2.11. *(Finite intersection property) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\cap K_\alpha$ is nonempty.*

Proof. (Proof by contradiction:) Suppose the arbitrary intersection is empty. Fix a member K_1 of $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^c$. Then the sets G_α form an open cover of K_1. K_1 is compact implies that there are finitely many indices $\alpha_1, \alpha_2, ...\alpha_N$ such that $K_1 \subset G_{\alpha_1} \cup ...G_{\alpha_n}$. But this mean that $K_1 \cap K_{\alpha_1} \cap ...K_{\alpha_n}$ is empty which is a contradiction. □

Corollary 2.12. *If K_n is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ for $n = 1, 2...$ then $\cap K_n$ is not empty.*

Theorem 2.13. *Let $\{I_n\}$ be a sequence of intervals (we can prove similarly for k-cells) in $\mathbb{R}^1$ such that $I_n \supset I_{n+1}$ for $n = 1, 2..$ then $\cap_n I_n$ is not empty.*

Proof. If $I_n = [a_n, b_n]$, let E be the set of all a_n. Then E is nonempty and bounded above by b_1. Let x be the sup of E. Since $a_n \le a_{m+n} \le b_{m+n} \le b_m$ for all positive integers m and n, we have $x \le b_m$ for all m. Since it is obvious that $a_m \le x$ we see that $x \in I_m$ for $m = 1, 2..$ □

Theorem 2.14. *Every k-cell is compact.*

Proof. Let I be a $k-$cell consisting of all points $x = (x_1, ...x_k)$ such that $a_j \le x_j \le b_j$. Put $\delta = [\sum_{j=1}^k (b_j - a_j)^2]^{\frac{1}{2}}$. Then $|x - y| \le \delta$ if $x, y \in I$.
Suppose, if there exists an open cover $\{G_\alpha\}$ of I which contains no finite subcover of I:
Put $c_j = (a_j + b_j)/2$. The intervals $[a_j, c_j]$ and $[c_j, b_j]$ then determine 2^k $k-$cells Q_i whose union is I. At least one of these sets Q_i call I_1 cannot be covered by any finite subcollection of $\{G_\alpha\}$ (otherwise I could be covered). We next subdivide I_1 and continue the process. We obtain a sequence $\{I_n\}$ with the following properties:

i_1 $I \supset I_1 \supset I_2 \supset ...$;
i_2 I_n is not covered by any finite subcollection of $\{G_\alpha\}$;
i_3 if $x, y \in I_n$ then $|x - y| \le 2^{-n}\delta$.

By (i_1) and the above theorem there is a point x^* which lies in every I_n. For some α, $x^* \in G_\alpha$. Since G_α is open, there exists $r > 0$ such that $|y - x*| < r$ implies that $y \in G_\alpha$. If n is so large that $2^{-n}\delta < r$ (there is such n because of archimedean) then (i_3) implies that $I_n \subset G_\alpha$ which contradicts (i_2). □

Theorem 2.15. *Heini-Borel Theorem: If a set E in $\mathbb{R}^k$ has one of the following three properties then it has the other two:*

1 *E is closed and bounded*
2 *E is compact*
3 *Every infinite subset of E has a limit point in E.*

*Proof*1 $\implies$ 2 Since E is closed and bounded, $E \subset I$ for some k-cell. Since closed subset of a compact set is compact 2 follows.

2 $\implies$ 3 (**Theorem:** If S is an infinite subset of a compact set E then S has a limit point in E.) If no point of E were a limit point of S then each $q \in E$ would have a neighborhood V_q which contains at most one point of S (namely, q if $q \in S$). It is clear that no finite subcollection of $\{V_q\}$ can cover S and the same is true of E since $S \subset E$. This contradicts the compactness of E. (*Why is it clear that there exists no finite subcollection of* $\{V_q\}$ *that can cover S?* Because if there exists finite subcollection $\{V_{q_1}, ...V_{q_N}\}$ that can cover S. Thus if $s \in S$, then $s \in \cup_{j=1}^N V_{q_j}$. For some j, $s \in V_{q_j}$ and by the choice of V_{q_j}, V_{q_j} has at the maximum one point of S, that is q_j: so, $s = q_j$. This implies every element in S is one of $\{q_1, ...q_N\}$. But S is infinite.)

3 $\implies$ 1 If E is not bounded, then E contains points x_n with $|x_n| > n$ $(n = 1, 2..)$. The set $S = \{x_n : n = 1, 2..\}$ is infinite and clearly has no limit point in $\mathbb{R}^k$ and hence has none in E. Thus E is bounded.
If E is not closed, then there is a point $x_0 \in \mathbb{R}^k$ which is a limit point of E but not a point of E. For $n = 1, 2..$ there are points $x_n \in E$ such that $|x_n - x_0| < 1/n$. Let S_1 be the set of these points x_n. Then S is infinite (otherwise $|x - x_0|$ would have a constant positive value for infinitely many n) S_1 has x_0 as a limit point and S has no other limit point in $\mathbb{R}^k$. For if $y \in \mathbb{R}^k$, $y \neq x_0$ then $|x_n - y| \geq |x_0 - y| - |x_n - x_0| \geq |x_0 - y| - \frac{1}{n} \geq \frac{1}{2}|x_0 - y|$ for all but finitely many n. This shows that y is not a limit point of S_1. (Since neighborhood of a limit point has infinitely many points). Contradicts our assumption 3. Hence E is closed.

□

Extras: How about general metric spaces? Which implies which? 2 <=> 3 But 1 => 2(or)3. Not the otherwise. Example given by L^2 functions-Space of square integrable function, which will be defined in the next chapter, if students are interested.

Theorem 2.16. *Weierstrass: Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

Proof. Being bounded the set E is a subset of a $k-$cell in $\mathbb{R}^k$. Every infinite subset of a compact set has limit point in that compact set. Hence the proof. □

Theorem 2.17. *Let P be a non empty perfect set in $\mathbb{R}^k$. Then P is uncountable.*

Proof. Since P has limit points P must be infinite. Suppose P is countable and denote the points of P by $x_1, x_2, ...$ We shall construct a sequence $\{V_n\}$ of neighborhoods, as follows:
Let V_1 be any neighborhood of x_1. If V_1 consists of all $y \in \mathbb{R}^k$ such that $|y - x_1| < r$, the closure $\overline{V}$ of V_1 is the set of all $y \in \mathbb{R}^k$ such that $|y - x_1| \leq r$.
Suppose V_n has been constructed so that $V_n \cap P$ is not empty. Since every point of P is a limit point of P there is a neighborhood V_{n+1} such that $(i)\overline{V}_{n+1} \subset V_n$, $(ii)x_n \notin \overline{V}_{n+1}$, $(iii)V_{n+1} \cap P$ is not empty. By (iii) V_{n+1} satisfies our induction hypothesis, and the construction can [proceed.
Put $K_n = \overline{V_n} \cap P$. Since $\overline{V}_n$ is closed and bounded, it is compact. Since $x_n \notin K_{n+1}$ no point of P lies in $\cap K_n$. Since $K_n \subset P$ this implies that $\cap K_n$ is empty. But by finite intersection property, it is a contradiction. □

Corollary 2.18. *Every interval is uncountable. In particular the set of all real numbers in uncountable.*

Theorem 2.19. *Cantor set is compact, nonempty, contains no segment and is perfect (and hence uncountable).*

Proof. Cantor set is the intersection of closed sets and hence closed. Also it is bounded. Thus the Cantor set is compact. Clearly nonempty. Any open segment (α, β) in $[0, 1]$ will have an open segment $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$ for $k, m \in \mathbb{Z}$ if $3^{-m} < (\beta - \alpha)/6$. However no segment of these form has a point in common with P. Thus Cantor set has no segment. To show that P is perfect it is enough to show that P contains no isolated point. Let $x \in P$ and let S be any segment containing x. Let I_n be that interval of E_n which contains x. Choose n large enough so that $I_n \subset S$. Let x_n be an endpoint of I_n such that $x_n \neq x$. It follows from the construction of the cantor set P that $x_n \in P$. Hence x is a limit point of P and P is perfect. □

End with the definition of connected set.

2.1. **Let us get back to sequence and series...** Apart from the following theorems (to be either done by the students or done in the extra classes) before starting continuity, we revise some more facts about sequences and series:

(1) Summation by parts and its application to say when a series is convergent
(2) Alternating series (Leibnitz)
(3) Product of series (Given as an assignment)
(4) Rearrangement of series
(5) Conditional convergence (converges but not absolutely), almost sure convergence (probabilistic way)
(6) A metric space in which every Cauchy sequence converges is said to be complete. (Seminars are based on this)
(7) Extras(if time permits): "relatively open and compact"

CARDINALITY: In set theory, an ordinal number, or ordinal, is one generalization of the concept of a natural number that is used to describe a way to arrange a collection of objects in order, one after another. Any finite collection of objects can be put in order just by the process of counting: labeling the objects with distinct whole numbers. The original definition of ordinal number: the order type of a well-ordering as the set of all well-orderings similar (order-isomorphic) to that well-ordering: in other words, an ordinal number is genuinely an equivalence class of well-ordered sets.

$\aleph_0$ (aleph-naught, also aleph zero or the German term Aleph-null) is the cardinality of the set of all natural numbers, and is an infinite cardinal. The set of all finite ordinals, called ω or ω_0 (where ω is the lowercase Greek letter omega), has cardinality $\aleph_0$. A set has cardinality $\aleph_0$ if and only if it is countably infinite, that is, there is a bijection (one-to-one correspondence) between it and the natural numbers.
$\aleph_1$ is the cardinality of the set of all countable ordinal numbers, called ω_1 or (sometimes) Ω. This ω_1 is itself an ordinal number larger than all countable ones, so it is an uncountable set. Therefore, $\aleph_1$ is distinct from $\aleph_0$. The definition of $\aleph_1$ implies (without the axiom of choice) that no cardinal number is between $\aleph_0$ and $\aleph_1$. If the axiom of choice (A choice function is a function f, defined on a collection X of nonempty sets, such that for every set A in X, f(A) is an element of A. Axiom of choice:For any set X of nonempty sets, there exists a choice function f defined on X) is used, it can be further proved that the class of cardinal numbers is totally ordered, and thus $\aleph_1$ is the second-smallest infinite cardinal number.

Theorem 2.20. *[Seminar] If $\{p_n\}$ is a sequence in a compact metric space X then some subsequence of $\{p_n\}$ converges to a point of X.*

Proof. Let E be the range of $\{p_n\}$. If E is finite then there is a $p \in E$ and a sequence $\{n_i\}$ with $n_1 < n_2 < ...$ such that $p_{n_1} = p_{n_2} ... = p$. The subsequence $\{p_{n_i}\}$ so obtained converges evidently to p. If E is infinite by (*infinite subset of a compact set has a limit point in the compact set.*) we know that E has a limit point $p \in X$. Choose n_1 so that $d(p_{n_1}, p) < 1$. Having chosen $n_1, ...n_{i-1}$ we see from *Neighborhood of a limit point contains infinitely many points* that there is an integer $n_i > n_{i-1}$ such that $d(p, p_{n_i}) < 1/i$. Then $\{p_{n_i}\}$ converges to p. □

Corollary 2.21 (Seminar-bolzano weierstrass)**.** *Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence. (Follows from the theorem* "There exists a subsequence of a sequence in a compact set that converges to a point in that set" *and Heini-Borel theorem)*

Theorem 2.22 (Seminar)**.** *If $\overline{E}$ is the closure of a set E in a metric space X then diam $\overline{E}$ =diam E. (diam E = sup $\{d(p,q) : p, q \in E\}$)*

Proof. Since $E \subset \overline{E}$ it is clear that diam $E \leq$ diam $\overline{E}$. Fix $\epsilon > 0$ and choose $p, q \in \overline{E}$. By the definition of $\overline{E}$ there are points $p', q' \in E$ such that $d(p, p') < \epsilon, d(q, q') < \epsilon$. Hence $d(p, q) \leq 2\epsilon + d(p'q')$ And thus diam $\overline{E} \leq 2\epsilon +$ diam E. □

Corollary 2.23 (seminar)**.** *If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ $(n = 1, 2...)$ and if $lim_{n\to\infty}$ diam $K_n = 0$, then $K = \cup_n K_n$ consists of exactly one point.*

Proof. K is not empty from **finite intersection property**. If K contains more than one point then diam $K > 0$. But for each n, $K_n \supset K$ so that diam $K_n \geq$ diam K. Contradiction.) □

Theorem 2.24 (Seminar and assignment)**.** *Recollect Cauchy criterion.*

(1) *In any metric space every convergent sequence is a Cauchy sequence (same as the proof we did in the previous chapter.)*
(2) *If X is compact and if $\{p_n\}$ is a Cauchy sequence in X then $\{p_n\}$ converges to some point of X.*

In $\mathbb{R}^k$ every Cauchy sequence converges, that is Euclidean space is complete.

Proof. (1) Suppose $p_n \to p$. That is, if $\epsilon > 0$ there there is an integer N such that $d(p_n, p) < \epsilon/2$ for all $n \geq N$. Hence

$$d(p_n, p_m) = \leq d(p_n, p) + d(p_m, p) \leq \epsilon$$

whenever $n \geq N$ and $m \geq N$. Hence $\{p_n\}$ is Cauchy.

(2) Let $\{p_n\}$ be a Cauchy sequence in the compact space X. For $N = 1, 2..$ let E_N be the set consisting of $p_N, p_{N+1}, ..$ Then $lim_{N\to\infty}$ diam $\overline{E}_N = 0$ by the above theorem. Being a closed subset of the compact space X each $\overline{E}_N$ is compact (Since every closed subset of a compact set is compact). Also $E_N \supset E_{N+1}$ so that $\overline{E}_N \supset \overline{E}_{N+1}$. Let $\epsilon > 0$. Since lim diam $E_N = 0$, there is an integer N_0 such that diam $\overline{E}_N < \epsilon$ for $N \geq N_0$. Since $p \in \overline{E}_N$, it follows that $d(p, q) < \epsilon$ for every $q \in \overline{E}_N$ and hence for every $q \in \overline{E}_N$. That is $d(p, p_n) < \epsilon$ if $n \geq N_0$.

Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Define E_N as above with x_j. For some N diam $E_N < 1$. The range of $\{x_n\}$ is the union of E_N and the finite set $\{x_1, ...x_{N-1}\}$. Hence $\{x_n\}$ is bounded. Since every bounded subset of $\mathbb{R}^k$ has compact closure in $\mathbb{R}^k$ (**Heini-Borel** theorem). Hence the proof. □

3. Continuity

(1) Let $S \subset \mathbb{R}^n$ and f be a function from S into $\mathbb{R}^m$. If a is a limit point of S then a point $v \in \mathbb{R}^m$ is the limit of f at a if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < d_n(x, a) < \delta \implies d_m(f(x), v) < \epsilon \ \forall x \in S.$$

We write $lim_{x \to a} f(x) = v$. (draw the graph: $f(a)$ might not be defined at all.)

(2) Let $S \subset \mathbb{R}^n$ and f be a function from S into $\mathbb{R}^m$. We say that f is continuous at $a \in S$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < d_n(x, a) < \delta \implies d_m(f(x), f(a)) < \epsilon \ \forall x \in S.$$

We write $lim_{x \to a} f(x) = v$. Moreover f is continuous on S if it is continuous at each point $a \in S$.

(3) Note that if a is an isolated point then f is always continuous at a.

(4) If f is not continuous at a we say that f is discontinuous at a.

(5) In particular we consider now functions on $\mathbb{R}$: Consider any point x such that $a \leq x < b$. We write $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) such that $(t_n \to x)$. Similarly we define $f(x-) = q$. It is clear that for any point $x \in (a, b)$, $lim_{t \to x} f(t)$ exists if and only if $f(x+) = f(x-) = lim_{t \to x} f(t)$.
   - (i) If f is discontinuous at a point x and if $f(x+)$ and $f(x-)$ exist then f is said to have a discontinuity of the first kind or a simple discontinuity at x. There are two ways a function can have a simple discontinuity: either $f(x+) \neq f(x-)$ or $f(x+) = f(x-) \neq f(x)$ (remark:removable singularity).
   - (ii) If either $f(x+)$ or $f(x-)$ does not exist, then we say that f is said to have a discontinuity of the second kind.

   f is said to be monotonically increasing on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. Similarly we define monotonically decreasing function.

EXAMPLES:

(1) discontinuity of second kind: $f(x)$ defined as 1 when $x \in \mathbb{Q}$ and 0 when $x \in \mathbb{R} \backslash \mathbb{Q}$.

(2) simple discontinuity at $x = 0$ of a function $f(x)$ defined on $(-3, 1)$ as $x + 2$ for $-3 < x < -2$, as $-x - 2$ for $-2 \leq x < 0$, and as $x + 2$ for $0 \leq x < 1$.

(3) simple discontinuity where neither $f(0+)$ nor $f(0-)$ exists: $sin(1/x)$ when $x \neq 0$ and 0 otherwise.

(4) $1/|x|$ is continuous on $\mathbb{R}^n \backslash \{0\}$ to $\mathbb{R}$.

relation between sequences and continuity...:

Theorem 3.1. *Let f be a function on $E \subset \mathbb{R}^m$ to $\mathbb{R}^n$ and p is a limit point of E. Then*

$$lim_{x \to p} f(x) = q \text{ if and only if } lim_{n \to \infty} f(p_n) = q,$$

for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $lim_{n \to \infty} p_n = p$. (Hint: (1) implies (2). supppose (1) does not hold, obtain sequence of radius $< 1/n$ from p to give a contradiction.)

Limits are unique.-Self read

Algebra of sequences and continuous(or) limit functions:-self read $f + g, f - g, fg, f\backslash g$ continuous(or)has limits whenever defined and when f, g are continuous (or) has limits.

Theorem 3.2. *-Self-read Composition of continuous function is continuous.*

Recall that for a function $f : A \underset{into}{\to} B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$ for $E \subset B$.

Theorem 3.3. *A mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous on $\mathbb{R}^m$ if and only if $f^{-1}(V)$ is open in $\mathbb{R}^m$ for every open set $V \subset \mathbb{R}^n$.*

Proof. Suppose f is continuous on $\mathbb{R}^m$.
Let V be open in $\mathbb{R}^n$. To show that every pointa neighborhood around p such that it is contained in $f^{-1}(V)$. Suppose $p \in f^{-1}(V)$, that is $f(p) \in V$. Since V is an open set, there exists an $\epsilon > 0$ such that

$$d_n(f(p), y) < \epsilon \implies y \in V. \tag{4}$$

Also f is continuous at p and hence there exists $\delta > 0$ such that

$$d_m(x, p) < \delta \implies d_n(f(x), f(p)) < \epsilon.$$

By equation (**??**),

$$d_m(x, p) < \delta \implies d_n(f(x), f(p)) < \epsilon \implies f(p) \in V, \therefore p \in f^{-1}(V).$$

Thus there exists $\delta > 0$ such that $d_m(x, p) < \delta$ implies $x \in f^{-1}(V)$.

Conversely: We assume $f^{-1}(V)$ is open for all open V. To prove f is continuous at p, for all p, that is for given $\epsilon > 0$, there exists $\delta_p > 0$ such that $d(p, y) < \delta \implies d(f(p), f(y)) < \epsilon$. Fix $p \in X$ and $\epsilon > 0$. Let V be the set of all $y \in \mathbb{R}^n$ such that $d_n(y, f(p)) < \epsilon$. Then V is open. By assumption, $f^{-1}(V)$ is open, that is, there exists $\delta > 0$ such that

$$d_m(p, x) < \delta \implies x \in f^{-1}(V) \therefore f(x) \in V,$$

By construction of V, $f(x) \in V \implies d_n(f(x), f(p)) < \epsilon$. Hence the proof. □

Corollary 3.4. *Continuous inverse image of a closed set is closed.*

Proof. Since we know that f is continuous if and only if inverse image of open sets V is open. Hence we prove $f^{-1}(V)$ is open for all open V if and only if $f^{-1}(C)$ is closed for all closed C.
Suppose we assume $f^{-1}(V)$ is open for all V is open. Let C be closed. To prove $f^{-1}(C)$ is closed, we prove that $(f^{-1}(C))^c$ is open. Since C is closed C^c is open. Hence $f^{-1}(C^c)$ is open. But

$$\begin{aligned} x \in f^{-1}(C^c) &\iff f(x) \in C^c \\ &\iff f(x) \notin C \\ &\iff x \notin f^{-1}(C) \\ &\iff x \in (f^{-1}(C))^c. \end{aligned}$$

Thus $f^{-1}(C^c) = (f^{-1}(C))^c$ is open.
Similarly since $f^{-1}(V^c) = (f^{-1}(V))^c$ we have the converse. □

Continuity and compactness:

(1) A mapping f of a set E into $\mathbb{R}^k$ is said to be **bounded** if there exists a real number M such that $|f(x)| \leq M$ for all $x \in E$.

(2) A mapping f of a metric space X into a metric space Y is uniformly continuous on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ for all $p, q \in X$ with $d_X(p, q) < \delta$. **Clearly uniformly continuous function is continuous.** Converse is not true: ex: $1/x$ on $(0, r)$.

Theorem 3.5. *Continuous image of a compact set $E \subset \mathbb{R}^m$ into a metric space Y is compact.*

Proof. To prove: $f(E)$ is compact, that is, if $\{U_\alpha\}_\alpha$ is an open cover of $f(E)$, we have to prove there exists a finite subcover of $f(E)$.

$$\begin{aligned} f(E) \subset \cup_\alpha U_\alpha & \\ x \in E \quad &\Longrightarrow \quad f(x) \in E \subset \cup_\alpha U_\alpha \\ &\Longrightarrow \quad f(x) \in U_\alpha \text{ for some } \alpha \\ &\Longrightarrow \quad x \in f^{-1}(U_\alpha) \text{ for some } \alpha \end{aligned}$$

that is, $E \subset \cup_\alpha f^{-1}(U_\alpha)$.

Since f is continuous, $f^{-1}(U_\alpha)$ is open for all α. Thus $\{f^{-1}(U_\alpha)\}_\alpha$ forms an open cover of E. Since E is compact, there exists finite subcover $\{f^{-1}(U_{\alpha_i})\}_{i=1}^N$. Thus $y \in f(E) \implies f^{-1}(y) \in E \implies f^{-1}(y) \in f^{-1}(U_{\alpha_i})$ for some i. Thus $\{U_{\alpha_i}\}_{i=1}^N$ forms a finite subcover of $f(E)$. □

Theorem 3.6. *Continuous mapping of a compact set into an Euclidean space is closed and bounded. (hence the function is bounded and the image is compact- using the previous theorem)-proof follows from Heini-Borel theorem*

In particular,

Theorem 3.7. *Suppose f is continuous real function on a compact metric space X.*

(1) ***Extreme Value Theorem:*** *Let $M = sup_{p \in X} f(p)$ and $m = inf_{p \in X} f(p)$. Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$. [M- least upper bound of the set of all numbers $f(p)$ where p ranges over X. Similarly, m greatest lower bound. In other words f attains maximum (at p) and minimum at q.]—Proof follows from "Let E be anon-empty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$."*

(2) *Then the function f is bounded (Proof: Continuous mapping of a compact set into an Euclidean space is closed and bounded. Hence the function is bounded.*

What if we drop the hypothesis that the metric space X is compact? The result does not hold. For example:

(1) E be a non-compact subset of metric space $\mathbb{R}$. Then E can be either "bounded and not closed" or "not bounded" (by Heini-Borel theorem).
Let E be bounded but not closed and $x_0 \notin E$ be a limit point of E. Consider $f : E \to X$ defined by $f(x) = \frac{1}{1+(x-x_0)^2}$. This is continuous (prove it!- Exercise). The function does not attain supremum in E disproving the

above result when the hypothesis of compactness is not satisfied.
Similarly if E is not bounded, consider $g(x) = x$ and conclude that the result does not hold.

(2) For E bounded but not closed consider $h_1(x) = \frac{1}{x-x_0}$ and when E is not bounded consider $h_2(x) = \frac{x^2}{1+x^2}$.

Theorem 3.8. *Suppose f is a continuous $1-1$ mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x$ for $x \in X$ is a continuous mapping of Y onto X.*

Proof. Let $g = f^{-1} : Y \to X$. To prove g is continuous, that is, to prove $g^{-1}(V) = f(V)$ is open for all open $V \subset X$. Since V is open, V^c is a closed subset of compact X. Thus V^c is compact. Since f is continuous, $f(V^c)$ is compact. Thus $f(V^c) = (f(V))^c = (g^{-1}(V))^c$ is closed. Hence the proof. □

Theorem 3.9. *Continuous function of a compact metric space X into a metric space Y is uniformly continuous on X.*

Proof. Let $f : X \to Y$ be a continuous function on a compact set X.
To prove that f is uniformly continuous on X, that is, to prove that for given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that $d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$.
Since f is continuous at $x \in X$ for all $\epsilon > 0$, there exists $\delta_{x,\epsilon} > 0$ such that

$$d(x, y) < \delta_{x,\epsilon} \implies d(f(x), f(y)) < \frac{\epsilon}{2} \tag{5}$$

Let $V_x = \{y : d(x, y) < \frac{\delta_{x,\epsilon}}{2}\}$. Then $X \subset \cup_x V_x$ and hence $\{V_x\}_x$ is an open cover of X. Since X is compact, there exists a finite subcover: $\{V_{x_i}\}_{i=1}^N$. Put $\delta_\epsilon = \frac{1}{3} \min_{1=1,..N} \{\delta_{x_i,\epsilon}\}$. Then for $p, q \in X \subset \cup_{i=1}^N V_{x_i}$, there exists i such that $q \in V_{x_i}$, thus $d(q, x_i) < \frac{\delta_{x_i}}{2}$. By (**??**),

$$d(f(q), f(x_i)) < \frac{\epsilon}{2} \tag{6}$$

Also, whenever $d(p, q) < \delta_\epsilon$, we have $d(p, x_i) \le d(p, q) + d(q, x_i) < \delta_\epsilon + \frac{\delta_{x_i}}{2} \le \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i}$. Hence whenever $d(p, q) < \delta_\epsilon$ by (**??**),

$$d(f(p), f(x_i)) < \frac{\epsilon}{2} \tag{7}$$

Thus whenever $d(p, q) < \delta_\epsilon$, by (**??**) and (**??**), we have

$$\begin{aligned} d(f(p), f(q)) &\le d(f(p), f(x_i)) + d(f(x_i), f(q)) \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

We cannot omit the compactness in this theorem: For example, For bounded but not closed E: Let $x_0 \notin E$ be a limit point of E. Consider $f : E \to X$ defined by $f(x) = \frac{1}{1+(x-x_0)^2}$. (Prove it is not uniformly continuous). **How about unbounded sets?** say real valued functions on set of integers?-Uniformly continuous. But $\psi(x) = x^2$ on $\mathbb{R}$ is continuous on $\mathbb{R}$ but not uniformly continuous.

Theorem 3.10. *Continuous image of a connected set is connected.*

Proof. Let $f : X \to Y$ be a continuous function and $E \subset X$ be a connected set. To prove: $f(E)$ is connected. Suppose $f(E)$ is not connected, that is, if there exists separated sets A and B ($A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$) such that $f(E) = A \cup B$. Consider $G = f^{-1}(A)$ and $H = f^{-1}(B)$.

$$\begin{aligned} x \in E &\iff f(x) \in f(E) = A \cup B \\ &\iff f(x) \in A or B \\ &\iff x \in f^{-1}(A) or f^{-1}(B) \end{aligned}$$

Thus $E = f^{-1}(A) \cup f^{-1}(B) = G \cup H$.

Also

$$\begin{aligned} x \in f^{-1}(A) &\iff f(x) \in A \subset \overline{A} \\ &\implies x \in f^{-1}(\overline{A}) \end{aligned}$$

Thus $G = f^{-1}(A) \subset f^{-1}(\overline{A})$.

Since $\overline{A}$ is closed and f is continuous, $f^{-1}(\overleftarrow{A})$ is closed set and contains$G$. But $\overline{G}$ is the smallest closed set that contains G. Hence $\overline{G} \subset f^{-1}(\overline{A})$. Thus

$$\overline{G} \cap H \subset f^{-1}(\overline{A}) \cap H = f^{-1}(\overline{A}) \cap f^{-1}(B). \tag{8}$$

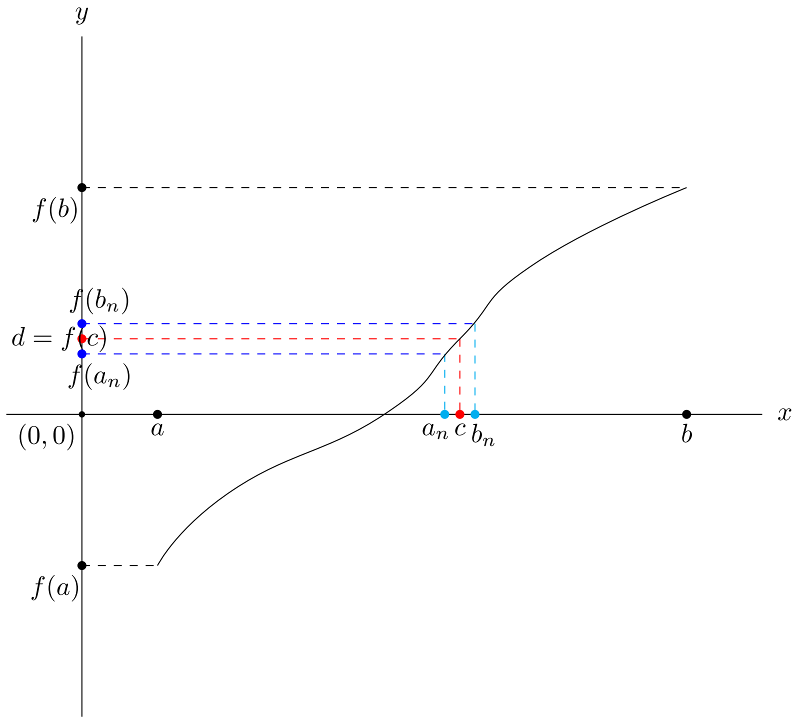
Also,

$$x \in f^{-1}(\overline{A}) \cap f^{-1}(B) \iff f(x) \in \overline{A} \cap B$$

Since $\overline{A} \cap B = \emptyset$, we have $f^{-1}(\overline{A}) \cap f^{-1}(B) = \emptyset$. By (**??**), $\overline{G} \cap H = \emptyset$. Similarly $G \cap \overline{H} = \emptyset$ and $E = G \cap H$. This is a contradiction to E being connected. Hence $f(E)$ is connected. □

Theorem 3.11. ***Intermediate value theorem:*** *If f is a continuous real-valued function on $[a, b]$ with $f(a) < d < f(b)$, then there exists a point $c \in (a, b)$ such that $f(c) = d$.*

Figure 1:



Proof. Define a subset $A \subset [a, b]$ by

$$A = \{x \in [a, b] : f(x) < d\}.$$

Since $a \in A$, A is non-empty. Also b is an upper bound for A. Thus there exists $c = \sup A$ and by Theorem **??** since $[a, b]$ is closed, $c \in [a, b]$. We claim that $f(c) = d$. Since c is the least upper bound for A, there is a sequence $(a_n) \in A$ such that $c - \frac{1}{n} < a_n < c$. So $c = \lim_{n\to\infty} a_n$. Therefore $f(a_n) < d$ for all $n \geq 1$ and f is continuous at c, that is

$$f(c) = \lim_{n\to\infty} f(a_n) \leq d. \tag{9}$$

That is, $c < b$. Choose any sequence $c < b_n \leq b$ such that $c = \lim_{n\to\infty} b_n$. Since c is the upper bound for A it follows that $b_n \notin A$ and so $f(b_n) \geq d$. Consequently

$$f(c) = \lim_{n\to\infty} f(b_n) \geq d. \tag{10}$$

From (**??**) and (**??**) we have

$$f(c) = d.$$

□

By Theorem **??**(Part 1) and Theorem **??**, we have the following corollary:

Corollary 3.12. *Continuous image of a closed interal is closed interval.*

Self read: Monotonicity and discontinuity.Self Read: Monotonic functions have no discontinuities of second kind. Let $f : (a, b) \to \mathbb{R}$ be a monotonic function on (a,b). Then the set of points of (a,b) at which f is discontinuous is at most countable. (Use (**??**) and conclude by finding a bijection between set of discontinuities and rational numbers) (*Hint:* A real valued function f on (a,b) is said to be monotonically increasing on (a,b) if for all $a < x < y < b \implies f(x) \leq f(y)$. Similarly we define monotonically decreasing and we call a function monotonic if it is either monotonically increasing or monotonically decreasing.)

Proof. A function f is said to have discontinuity of second kind at x if either $f(x+)$ or $f(x-)$ does not exist. So to prove that f cannot have discontinuities of second kind we prove that for all $x \in (a,b)$ both $f(x+)$ and $f(x-)$ exist.
Let us assume f is monotonically increasing function.(Analogously we can prove for monotonically decreasing function.)
That is, for fixed $x \in (a,b)$,

$$
\begin{aligned}
&a < t < x < b \implies f(t) < f(x) \\
&\{f(t) : a < t < x\} \text{ is bounded above by } f(x) \\
&\text{and } A_x = \sup_{a<t<x} \{f(t) : a < t < x\} \leq f(x).
\end{aligned}
$$

We have to say that $f(x-)$ exists. So we prove that $A_x = f(x-)$, that is, for all sequences $\{t_n\}$ in (a,x) such that $t_n \to x$, we have to prove that $f(t_n) \to A_x$. Let $\epsilon > 0$ be given. Then we have to find N such that $|f(t_n) - A_x| < \epsilon$ for all $n \geq N$.
By definition of A_x, for given $\epsilon > 0$, since $A_x - \epsilon < A_x$, there exists t such that $a < t < x$ with $A_x - \epsilon < f(t) < A_x$. We fix $\delta = x - t > 0$. Then we have

$$A_x - \epsilon < f(x - \delta) < A_x. \tag{11}$$

Now, since $t_n \to x$ and $t_n < x$ there exists N such that for all $n \geq N$, $x - t_n < \delta$, that is, $x - \delta < t_n < x < b$. Since f is monotonically increasing, $f(x-\delta) \leq f(t_n) < f(x)$.
By (**??**), we have $A_x - \epsilon < f(x - \delta) < f(t_n)$, that is, $A_x - f(t_n) < \epsilon$ for all $n \geq N$.
Since A_x is the supremum over $t < x$, $f(t_n) \leq A_x$ for all n. Thus $|A_x - f(t_n)| < \epsilon$ for all $n \geq N$.
Similarly we prove that $f(x+)$ exists. Hence the proof. □

Conclude that: If f is monotonically increasing and $a < x < y < b$ then

$$f(x+) \leq f(y-). \tag{12}$$

Extras: After defining continuity: spaces of continuous functions on $[0,1]$ and define L^p metric to remark on Heini Borel theorem on other metric spaces (closed and bounded (disc) but not compact)
Approximate continuity: Generalization of limit. A function $f : \mathbb{R}^k \to \mathbb{R}^m$ is said to have an approximate limit y at a point x if there exists a set F that has a density (to be studied later in measure theory.. to understand it just means that — the edge of the set, that is the set of points in F whose neighborhood is partially in F and partially outside of F is negligible) at the point such that if x_n is a sequence in F that converges towards x then $f(x_n)$ converges to y. Clearly, f has an ordinary limit at x then it also has an approximate limit with the same value.

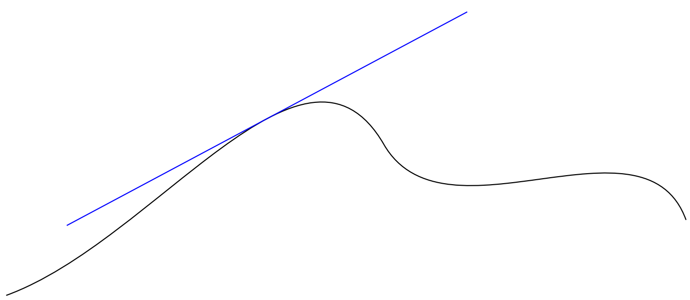
4. Differentiation

Motivation: The key point to study about the derivative of a function is since there is a connection between the derivative and the average slope; and hence it

deduces the properties of f from the derivative.
A real-valued function $f\ [a,b] \to \mathbb{R}$ is **differentiable** at $x_0 \in (a,b)$ if

$$\underset{h\to 0}{lim}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{x_0\to x}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \text{ exists.}$$

For a function on closed interval the it is differentiable at the end points if the appropriate one-sided limit exists. When the function is differentiable at all points of the interval I we say that is differentiable on I.
When f is differentiable at x_0 we define the tangent line to f at x_0 to be the affine function (*affine functions* $f(x) = l(x) + c$ *where* $l(x)$ *a linear function - Seen in Linear algebra:*) $T(x) = f(x_0) + f'(x_0)(x - x_0)$. Note that if f is differentiable at a point then the tangent line passes through the point $(x_0, f(x_0))$ with slope $f'(x_0)$.



Theorem 4.1. ***differentiability implies continuity*** *If f is differentiable at x_0 then it is continuous at x_0.*

Proof. We have

$$f(t) - f(x) = \frac{f(t)-f(x)}{t-x}.(t-x)$$

whenever $t \neq x$. We know that $\lim(f_1f_2)(x) = \lim f_1(x)\lim f_2(x)$ (was a homework). Hence

$$\lim_{t\to x} f(t) - f(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x}.\lim_{t\to x}(t-x) = f'(x) = 0$$

Thus $\lim_{t\to x} f(t) = f(x)$. □

Note that *'locally'* the slope of the tangent line equals the slope of the function. We see that the function is **'approximated'** by tangent lines:

Theorem 4.2. *Let f be a function on $[a,b]$ that is differentiable at x_0. Let $T(x)$ be the tangent line to f at x_0. Then T is the unique affine (line) function with property that* $\underset{x\to x_0}{lim}\frac{f(x)-T(x)}{x-x_0} = 0$.

Proof. It is clear that for $T(x) = f(x_0) + f'(x_0)(x - x_0)$, $\underset{x\to x_0}{lim}\frac{f(x)-T(x)}{x-x_0} = 0$. We prove the uniqueness. Suppose if there is another affine (line) function $L(x) = m.x + c$ with slope m such that $\underset{x\to x_0}{lim}\frac{f(x)-mx-c}{x-x_0} = 0$, then since f and L both are continuous function

$$\begin{aligned} f(x_0) - mx_0 - c &= \lim_{x\to x_0} f(x) - mx - c \\ &= \lim_{x\to x_0}(x-x_0)\frac{f(x)-mx-c}{x-x_0} = 0. \end{aligned}$$

Thus $L(x_0) = mx_0 + c = f'(x_0)$ and $L(x) = m(x - x_0) + f'(x_0)$. Thus

$$\begin{aligned} m = L'(x_0) &= \lim_{h\to 0} \frac{L(x_0+h) - L(x_0)}{h} \\ &= \lim_{h\to 0} \frac{L(x_0+h) - f(x_0+h)}{h} + \frac{f(x_0+h) - f(x_0)}{h} \\ &= 0 + f'(x_0) \end{aligned}$$

Thus $L(x) = f'(x_0)(x - x_0) + f'(x_0) = T(x)$. □

Note the following:

Corollary 4.3. *If f is a function on (a, b) and $x_0 \in (a, b)$ then the following are equivalent:*

1. *f is differentiable at x_0*
2. *there is a function $\phi(x)$ on (a, b) such that $f(x) = f(x_0) + \phi(x)(x - x_0)$ and $\lim_{x\to x_0} \phi(x)$ exists.*

How about the converse? **Not true.** Consider *(the standard example)* $f(x) = |x|$ on any interval that does not contain 0. When we say that a function is not differentiable, we say that the limit in the definition of differentiable functions is not exiting. In the example the right limit in the definition or the right derivative is 1 and the left derivative is -1.

Algebra of differentiable functions and examples of continuous function but not differentiable function

Theorem 4.4. ***Chain rule*** or differentiation of composition of functions: *Suppose that f is defined on $[a, b]$ and has range contained in $[c, d]$. Let g be defined on $[c, d]$. Suppose that f is differentiable at $x_0 \in [a, b]$ and g is differentiable at $f(x_0)$. Then $h(x) = g(f(x))$ is defined on $[a, b]$ and $h'(x_0) = g'(f(x_0))f'(x_0)$.* Hint: use Corollary **??**

differentiable except at one point but continuous everywhere: $f(x) = xsin(1/x)$ for $x \neq 0$ and 0 for $x = 0$. (continuous at 0 since $|sin(1/x)| \leq 1$ and $x \to 0$ for x very small)

A real function f defined on S is said to have **local maximum** at a point x_0 if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in S$ with $d(x, x_0) < \delta$.

Theorem 4.5. ***Fermat's theorem:*** *Let f be a continuous function on an interval $[a, b]$ that takes its maximum or minimum value at a point x_0. Then*

1. *x_0 is an endpoint that is $x_0 = a$ or $x_0 = b$ (or)*
2. *f is not differentiable at x_0 (or)*
3. *f is differentiable at x_0 and $f'(x_0) = 0$.*

In other words, *if real function f defined on (a, b) has a local maximum at a point $x_0 \in (a, b)$ and if $f'(x_0)$ exists then $f'(x_0) = 0$*

Proof. Suppose the first two options are not applicable. Since x_0 is an interior point, that is there exists $\delta_1 > 0$ such that $a < x_0 - \delta_1 < x < x_0 + \delta_1 < b$ and which f is differentiable at x_0. Since f has a local maximum at x_0 there exists $\delta_2 > 0$

such that $f(x) < f(x_0)$ for all $|x - x_0| < \delta_2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then for all $x_0 - \delta < t < x_0$, we have

$$\frac{f(t) - f(x_0)}{t - x_0} \geq 0$$

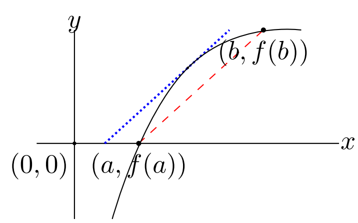
Letting $t \to x_0$, we have the right derivative of f at x_0 greater than or equal to zero. Since we assumed that $f'(x_0)$ exists, we have $f'(x_0) \geq 0$. Similarly for $x_0 < t < x_0 + \delta$, we have

$$\frac{f(t) - f(x_0)}{t - x_0} \leq 0$$

Letting $t \to x_0$, we have the left derivative of f at x_0 less than or equal to zero. Again since $f'(x_0)$ exists, we have $f'(x_0) \leq 0$. Thus $f'(x_0) = 0$. □

Theorem 4.6. ***Rolle's Theorem:*** *Suppose that f is a function that is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b) = 0$. Then there is a point $c \in (a, b)$ such that $f'(c) = 0$. (**Differentiability at every interior point of $[a, b]$ is necessary:** Example: $f(x) = 1 - |x|$ on $[-1, 1]$. It does not have any point at which $f'(x) = 0$. f is differentiable at every point except the point $x = 0$ at which the maximum occurs.)*

Proof. If the maximum and minimum value of f are both 0 then $f(c) = 0$ for all $c \in (a, b)$. Hence $f'(c) = 0$ for all $c \in (a, b)$. So without loss of generality, we assume that the maximum is greater than zero (similarly we can prove if the minimum is zero by considering the function $g = -f$.)
By the Extreme value theorem **??** there is a point c at which f attains maximum. Since the maximum value is greater than 0, c is an interior point. By Fermat's theorem **??**, we have $f'(c) = 0$. □



Theorem 4.7. *If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b) then there is a point $x \in (a, b)$ at which*

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Note that the differentiability is not reqired at end points.

Proof. Let $h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a))]f(t)$ for $a \leq t \leq b$. Then h is continuous on $[a, b]$ and differentiable on (a, b) and $h(a) = h(b)$. To prove the theorem we have to show that $h'(x) = 0$ for some $x \in (a, b)$. If h is constant then the theorem is proved. Suppose h is not constant and $h(t) > h(a)$ for some $t \in (a, b)$ (we can run the following argument by replacing maximum with minimum). By extreme value theorem **??**, h attains maximum at a point $x \in [a, b]$. Since $h(a) = h(b)$, we have $x \in (a, b)$. By Fermat's theorem **??**, we have $h'(x) = 0$. □

Theorem 4.8. ***Mean value theorem:*** *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b) then there is a point $x \in (a, b)$ at which $f(b) - f(a) = (b - a)f'(x)$. (*Proof: Apply previous theorem with $g(x) = x$.*)*

Corollary 4.9. *If f is differentiable in (a, b),*

(1) *if $f'(x) \geq 0$ for all $x \in (a, b)$ then f is monotonically increasing.*
(2) *if $f'(x) = 0$ for all $x \in (a, b)$ then f is constant.*
(3) *if $f'(x) \leq 0$ for all $x \in (a, b)$ then f is monotonically decreasing.*

Theorem 4.10. ***Darboux theorem:*** *Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.*
A similar result holds for $f'(a) > f'(b)$.

Proof. Let $g(t) = f(t) - \lambda t$. Then $g'(a) < 0$ and $g'(b) > 0$ so that there exists $t_1, t_2 \in (a, b)$ such that $g(t_1) < g(a)$ and $g(t_2) < g(b)$. Thus by the extreme value theorem **??** g attains minimum at a point $x \in (a, b)$. By Fermat's theorem **??** $g'(x) = 0$. Hence $f'(x) = \lambda$. □

Corollary 4.11. *If f is differentiable on $[a, b]$ then f' cannot have any simple discontinuities on $[a, b]$.*

5. Integration

Refer: Rudin's baby book.
Throughout this section, we discussed $f \in \mathcal{R}(\alpha)$ where α is monotonically increasing function. Now if we have $\alpha = \alpha_1 - \alpha_2$, where α_1 and α_2 are monotonic increasing functions, then both $\int f d\alpha_1$ and $\int f d\alpha_2$ exist. Hence by the properties of integrals, we can write $\int f d\alpha = \int f d\alpha_1 - \int f d\alpha_2$. Hence the whole theory goes through for $f \in \mathcal{R}(\alpha)$ where α is written as the difference of the two monotonic increasing functions. In the next section, we study about functions α which are written as the difference of monotonic increasing functions (called Bounded variation).

6. Bounded variation and rectifiablity-short notes

Refer Kreuger-Kannan's book for more:

Motivation: We have studied continuous functions which "behave smoothly". Similarly differentiable functions "behave better than" continuous functions. To be precise set of all differentiable functions is a subset of all continuous functions. Functions of bounded variation are functions may not be continuous everywhere, "mostly differentiable". We will see how functions of bounded variation are precisely difference of two monotone functions, thereby they "behave well". **Recall:** Let $f : [a, b] \to \mathbb{R}$ be a given function.

- f is said to be (strictly) increasing on $[a, b]$ if for every $x, y \in [a, b]$, $x < y \implies f(x)(<) \leq f(y)$.
- f is said to be (strictly) decreasing on $[a, b]$ if for every $x, y \in [a, b]$, $x < y \implies f(x)(>) \geq f(y)$.
- f is said to be (strictly) monotone if it is either an (strictly) increasing function or (strictly) decreasing function on $[a, b]$.

Definition 6.1. *Let $f : [a,b] \to \mathbb{C}$ be given. Given any finite partition*

$$\Gamma = \{a = x_0 < x_1 < ... < x_n = b\}$$

of $[a,b]$ set $S_\Gamma = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$. The variation of f over $[a,b]$ is

$$V[f;a,b] = \sup\{S_\Gamma : \Gamma \text{ is a partition of } [a,b]\}.$$

The function f has bounded variation on $[a,b]$ if $V[f;a,b] < \infty$. We set

$$BV[a,b] = \{f : [a,b] \to \mathbb{C} : f \text{ has bounded variation on } [a,b]\}.$$

Note that the variation of f 'represents' the total vertical distance traveled by a particle that moves along the graph of f from $(a, f(a))$ to $(b, f(b))$.
The **variation** $f : \mathbb{R} \to \mathbb{R}$ is $V[f;\mathbb{R}] = \sup_{a<b} V[f;a,b]$. We say that f has bounded variation if $V[f;\mathbb{R}] < \infty$.
Recall that a function $f : [a,b] \to \mathbb{R}$ is said to satisfy Lipshitz condition if there exists a constant $M > 0$ such that for every $x, y \in [a,b]$ we have $|f(x) - f(y)| \leq M|x - y|$. Lipshitz functions are examples of functions on $[a,b]$ that have bounded variation.
Notation: $x^+ = max\{x, 0\}$ and $x^- = max\{x, 0\}$.
Let $f : [a,b] \to \mathbb{R}$ be given. The **positive variation** of f on $[a,b]$ is

$$V^+[f:a,b] = \sup\{\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]^+ : \Gamma = \{a = x_0 < ...x_n = b\} \text{ is a partition of } [a,b]\}$$

and the **negative variation** is

$$V^-[f:a,b] = \sup\{\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]^- : \Gamma = \{a = x_0 < ...x_n = b\} \text{ is a partition of } [a,b]\}.$$

Lemma 6.2. *For any $f : [a,b] \to \mathbb{R}$ we have $V^+[f:a,b] + V^-[f;a,b] = V[f;a,b]$. Further, if any one of $V[f;a,b]$, $V^+[f;a,b]$, or $V^-[f;a,b]$ is finite, then they are all finite and in this case we also have $V^+[f;a,b] - V^-[f;a,b] = f(b) - f(a)$.*

Lemma 6.3. *For any $f : [a,b] \to \mathbb{R}$ we have $V^+[f;a,b] + V^-[f;a,b] = V[f;a,b]$*

Theorem 6.4. *Jordan Decomposition: Let $f : [a,b] \to \mathbb{R}$. $V[f;a,b] < \infty$ if and only if "$f = f_1 - f_2$ where f_1 and f_2 are monotonically increasing functions".*

Some of the results:

(1) If $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and if there exists $M > 0$ such that $|f'(x)| \leq M$ on $[a,b]$ then $V[f;a,b] \leq M(b-a) < \infty$.
(2) If f is continuously differentiable on $[a,b]$ then $V[f;a,b] = \int_a^b |f'(x)|dx$.
(3) Let $V[f;a,b] < \infty$. Then $v(x) = V[f;a,x]$ is continuous at a point c if and only if f is continuous at c. (In fact, if $V[f;a,b] < \infty$ then we have for any $x \in (a,b)$, $\lim_{h\to 0} v(x+h) - v(x) = |\lim_{h\to 0} f(x+h) - f(x)|$ and $v(x) - \lim_{h\to 0} v(x-h) = |f(x) - \lim_{h\to 0} f(x-h)|$)
(4) If f has a continuous derivative on $[a,b]$ then the function $v(x) = V[f;a,x]$ is differentiable and has a continuous derivative on $[a,b]$.
(5) If $V[f;a,b] < \infty$ and f is continuous on $[a,b]$ then $V[f;a,b] = \lim_{h\to 0} \frac{1}{h}\int_a^{b-h} |f(x) - f(x+h)|dx$

6.1. **absolute continuity and rectifiablity. To reconstruct a function from its dertivative we study absolute continuity:** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. f is said to be absolutely continuous on $[a, b]$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\left| \sum_{k=1}^{n} f(b_k) - f(a_k) \right| < \epsilon$ for any $a \leq a_1 < b_1 \leq a_2 < b_2 \leq ... a_n < b_n \leq b$ for which $\sum_{k=1}^{n} (b_k - a_k) < \delta$.

Note that

(1) absolute continuity implies continuity
(2) absolute continuity implies bounded variation

Theorem 6.5. *Let $f : [a, b] \to \mathbb{R}$. Then f is absolutely continuous if and only if there exists $f \in L^1[a, b]$ such that $f(x) - f(a) = \int_a^x h(t)dt$. It will follow that $h = f'$.*

"Lebesgue points" are useful for "approximations". We can use Lebesgue points to improve the fact $h = f'$ in the above theorem:

If $\lim\limits_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt = 0$ at the point x then x is called a Lebesgue point of the function f.

(1) If $f \in \mathcal{R}$ then almost every point of $[a, b]$ is a Lebesgue point of f, that is the length of the set $\{x \in [a, b] : x \text{ is not a Lebesgue point of } f\}$ is zero.
(2) Suppose $f \in \mathcal{R}$ on $[a, b]$. If f is continuous at x then x is a Lebsgue point of f.
(3) If f is differentiable and f' is bounded then f is absolutely continuous.

Nice exercise to see when continuity can be replaced with absolute continuity in the theorems we discussed in "Integration".

Rectifiability: Suppose $f : [a, b] \to \mathbb{R}^k$ where $f(t) = (f_1(t), f_2(t), ... f_k(t))$. Assume f_j for all $j = 1, ..k$ are continuous functions. Then we call the domain of f in $\mathbb{R}^k$ a curve. If the length of the curve is finite we say that the curve is rectifiable. The length of the curve is defined by

$$\sup\{\sum_{k=1}^{m} \|f(t_k) - f(t_{k-1})\|\}$$

(where the supremum is taken over all the partitions $P = \{a = t_0, t_1, ... t_m = b\}$ of $[a, b]$). In fact f is rectifiable if and only if each f_j $(j = 1, ..k)$ is a bounded variation.

7. Sequences and series of functions

Rudin's Baby book

References

[(Rud)Rudin] [Rud] Rudin, W. "Principles of Mathematical Analysis" *3rd Edition*(1976)

[(Dav)Davidson] [Dav] Davidson, Kenneth R., and Donsing, Allan P., "Real analysis and Applications Theory in Practice" *Springer*(2010)

[(KK)Krueger] [KK] Krueger, C. K., Kannan, R. "Advanced Analysis on the Real line" *Springer*(1996)

[(Bart)Bartley] [Bart] Bartley, Robert G., and Sherbert, Donald R. "Introduction to Real analysis" *4th Edition*(2014)

[(Kra)Krantz] [Kra] Krantz, Steven G. "A handbook of real variables" *Birkhauser*(2004)

[(T)Tao] [T] Tao, Terrance "Analysis I and II" *Hindustan Book agency*(2006)

[(Bart)Bartley] [Bart] Bartley, Robert G., and Sherbert, Donald R. "Introduction to Real analysis" *4th Edition*(2014)