ADVANCED LINEAR ALGEBRA - MTH 311

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Contents

- This course forms a basic step towards abstract algebra
  -∗- Those who are interested more towards algebraic concepts, like, properties of $SL(n)$ (-$n \times n$ matrices with determinant 1), the decomposition theorems are the basics, and you can look at more motivation like *Iwasawa decomposition*-see Appendix 2 in [4].
- This course is also intimately related to geometry. Hence this course equally concentrates on geometric point of view as well. If interested in more geometric view, please refer to Chapter 3 from [4] and chapter 12 can be read by the end of the course as a winter project.

1. Vector Spaces

Definition of Vector spaces
- criteria to check subspaces
- what are bases and dimension?
— Hamel - standard basis
- some examples including finite dimensional and infinite dimensional vector spaces
— also recall how the vector spaces behave with subfields.
- recall not every infinite dimensional vector spaces have explicit bases

Thought! - examples like spaces of continuous functions does not get benefited by basis, that is, by expressing finite linear combination of elements – There is no explicit basis of this space. When you are looking at a point you would want the continuous function to be written in terms of same 'nice' functions. However, with finite linear combination we might have any sort of functions upto our expectations. So this kind of algebraic (or) Hamel basis are not useful. In applications, with the help of various topologies the notion of a Schauder basis (-you will study later -*not in this course!*) becomes more useful.

- **important example(s):**
—— space of all polynomials over the field (See subsection 7.2)
——— and Differential equation point of view: Solution space of a system of homogeneous linear equations (see subsection 7.1).
—— self adjoint matrices is NOT a subspace of complex square matrices.

NOTES: (Refer [2] - Chapter 2 Sections: 2.1-2.3)

2. Linear Transformation

How are we going to relate matrices with real examples. Differential equations are one of the main important way/technique to study physical bodies. Similarly geometry plays very vital role. How are we going to deal geometry and differential equations via matrices? What is the analogue for infinite dimension?

- What is a linear transformation?
——— examples of linear transformations on finite and infinite dimensional vector spaces
- rank - nullity theorem – null space - hyperspace
——— applications in solving system of linear equations
- the algebra of linear transformations, invertible linear transformations
- isomorphism – any finite dimensional vector space over the field is isomorphically F^n
- matrix of a linear transformation
- change of basis
——— examples of different finite dimensional vector space and representing the linear transformations on them as matrices.
- linear functionals
——— important examples of linear functionals on both finite and infinite dimensional vector space
- annihilator of a subspace, dual space - transpose of a linear transformation - double dual, canonical isomorphism between a vector space and its double dual
- applications of annihilator, null space, dual space in picturizing lower dimensional lines, planes and so on in Euclidean space, various geometrical and linear equation problems

NOTES: (Refer [2] Chapter 3 Sections: 3.1-3.7)

3. Triangulation and diagonalization

We would like to analyse linear operators... to make them simple... we know that triangular matrices and diagonalizable matrices are better to understand... similarly can we simplify linear operators?:

one way of studying diagonalizable operator: Characteristic polynomials
Refer [2] Chapter 6 - Section 6.2
——— Characteristic values, eigenvectors, eigenspace
——— importance via diagonalizable linear operator - equivalent notions of diagonalizable operator (in terms of characteristic polynomial, dimensions of eigen spaces)
——— examples of how to diagonalize via linear operator
another way of studying diagonalizable operator: minimal polynomial
(Refer to [1]
——— what are annihilating polynomials, minimal polynomial?
——— importance of studying them
——— characterization of diagonalizable operator using the minimal polynomial
——— Cayley-Hamilton theorem
More tools for triangulation and diagonalization:
——— invariant subspaces, (Refer to Section 6.4 in [2]
——— simultaneous triangulation, simultaneous diagonalization(Refer to Section 6.5 in [2]

Triangulizable does not imply diagonalizable. See the example 4.1

Definition 3.1. *Recall from Section 6.4 in* [2] *the definition of invariant subspaces. Let dim* $(V, F) = n$, $T \in L(V)$, W *be a proper subspace of* V *which is* T*-invariant. The* T*-conductor polynomial into* W *of* $v \notin W$ *is the monic polynomial* $\mathfrak{g}$ *of the lowest degree for which* $\mathfrak{g}(T)v \in W$. *The* T*-conductor polynomial into* W *is the monic polynomial* $\mathfrak{p}$ *of the lowest degree for which* $\mathfrak{p}(T)(v) \in W \ \forall v \in V$.

$\mathfrak{g}|\mathfrak{p}$. If $W = \{0\}$, then $\mathfrak{p} \equiv \mathfrak{m}$

4. Primary sum (Jordan) decomposition (D-N)

Now that we know how to diagonalize, we would like to know how to approach any linear transformation and any vector space in a 'simplified' way?

- direct sum decompositions, projections, invariant direct sums, primary decomposition theorem, nilpotent operators, S-N decomposition.

- Any diagonalizable matrix (direct sum) decompose the vector space into 'nice' T-invariant subspaces.
- Consider the example 4.1 to see how decomposition (that we are going to discuss) plays role in a simple matrix.

NOTES: (Refer [2] Chapter 6 Sections: 6.6-6.8)

Theorem 4.1. *Primary sum decomposition theorem: (See Theorem 12 and 13 - section 6.8* [2]*)*
Let $T \in L(V)$ where dim $V < \infty$. Let the minimal polynomial for T be $\mathfrak{m}(x) = p_1^{r_1}(x) \cdots p_k^{r_k}(x)$ where p_i's are distinct irreducible (prime) monic polynomials over the field F and the r_i are positive integers. Then

(i) $V = N_{p_1^{r_1}(T)} \bigoplus \cdots \bigoplus N_{p_k^{r_k}(T)}$
(ii) *each $N_{p_i^{r_i}(T)}$ is $T-$invariant*
(iii) *if T_i is operator induced on $N_{p_i^{r_i}(T)}$ by T then the minimal polynomial for T_i is $p_i^{r_i}$.*
(iv) *If each p_i is linear, then $T = D + N$ where the diagonalizable operator D and the nilpotent operator N on V are uniquely determined such that $DN = ND$. This is Jordan decomposition*

Apart from the following example, also refer to the subsection 7.1.1.

4.1. **Example: Triangulable but not diagonalizable matrix.** Consider the matrix A given by an operator $T \in L(V)$ with respect some basis $\mathcal{B}$ of $V = \mathbb{R}^3$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Note the following:

- The characteristic polynomial of this operator T is $\mathfrak{c}(x) = (x-1)^2(x-4)$.

- Note that the matrices of $(T-I)$, $(T-I)^2$ and $T-4I$ with respect to the standard basis are respectively given as

$$(A-I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad (A-I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\text{and } (A-4I) = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

— This gives $N_{(T-I)^2} =< (1,0,0),(0,1,0) >$ and $N_{T-4I} =< (1,3,9) >$.

— In fact, the matrix of T with respect to the new basis from the above subspaces, that is with respect to $\mathcal{B} = \{(1,0,0),(0,1,0),(1,3,9)\}$ is

$$\left(\begin{array}{cc|c} 1 & 1 & \multirow{2}{*}{0} \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & 4 \end{array}\right)$$

Note that although the given matrix is not diagonalizable, it is similar to diagonally-block matrix. We see the importance of this form in this chapter.

- The eigenspace N_{T-I} (the null space of $T-I$) corresponding to the eigenvalue 1 has dimension 1 **which is not equal to the multiplicity of** $(x-1)$, that is 2. This via Theorem on characterization of diagonlaization says that this operator is **not diagonalizable**. Check that $\mathfrak{c}(T) \equiv 0$. Two ways to check:
  — Check with the matrix that $(A-I_3)^2(A-4I_3) = ([T]-I_3)^2([T]-4I_3) = 0$ and hence by the representation of the operator by a matrix, the operator $(T-I)^2(T-4)(v_i) = \sum A_{ji}v_j = \sum 0v_i = 0$ for all $v_i \in \mathcal{B}$. Thus $\mathfrak{c}(T)(v) \equiv (T-I)^2(T-4)(v) = 0$ for all $v \in V$.
  — another way to check: Find out the matrix representation of $(T-I)^2$. Check that nullspace $N_{(T-I)^2}$ is 2 dimensional and similarly the eigen space N_{T-4I} corresponding to the eigen value 4 is 1. Check that the vectors in $N_{(T-I)^2}$ and N_{T-4I} are linearly independent. Since these two has 0 as the only common element, the subspace formed by spanning both the eigen spaces is of dimension 3. Thus $\mathfrak{c}(T) \equiv 0$.
- **minimal polynomial**:
  By the definition of $\mathfrak{m}$, $\mathfrak{m}|\mathfrak{c}$. Let $m_1(x) = (x-1)(x-4)$. by the property of minimal polynomial , $(x-1)(x-4)$ should divide minimal polynomial. But $(T-I)(T-4I) \not\equiv 0$ (since eigenspaces of distinct eigen values have zero as the only common element and $dim\ N_{T-I} = 1 = dim\ N_{T-4I}$, but $dim\ V = 3$, we have a non-zero vector $v \notin N_{T-I} \cup N_{T-4I}$ such that $m_1(T)v \neq 0$, that is $m_1(T) \not\equiv 0$).
  Hence $\mathfrak{m} = \mathfrak{c}$.
- Hence we have $V = N_{(T-I)^2} \bigoplus N_{T-4I}$.
  —— (check why is it direct sum?)-apply **Primary decomposition** - Theorem 4.1.
  —— Both the subspaces $W_1 = N_{(T-I)^2}$ and $W = N_{T-4I}$ are T-Invariant?
  —— What happens to $T_i = T|_{W_i}$ the restriction of T on W_i?
- Note that

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} = D + N$$

where

$$D = \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \text{ and } \quad N = \begin{pmatrix} 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is obtained by **primary sum decomposition** -Theorem 4.1, where D is diagonalizable and N is nilpotent.

5. Rational and Jordan forms

We continue to view a vector space in a 'simplified' way for a fixed linear operator. characteristic and minimal polynomials, in general, do not convey how two matrices are similar. We see a characterization of similar matrices.

5.1. **Order of a vector:** We say a particular polynomial as the order of a vector:

Lemma 5.1. *Let V be a finite dimensional vector space over F and $T \in L(V)$. Let $v \in V$ be a non-zero vector. There exists a unique non-zero monic polynomial $\mathfrak{m}_v$ in $F[x]-$space of all polynomials over F such that $\mathfrak{m}_v(T)(v) = 0$ and $\mathfrak{m}_v | g$ for every $g \in F[x]$ such that $g(T)(v) = 0$.*

Proof. We outline the proof of existence. The proof of uniqueness is left as an exercise.
Since V is finite dimension, there exists a positive integer k such that $\{v, Tv, T^2v, \cdots, T^{k-1}v\}$ are linearly independent and $\{v, Tv, T^2v, \cdots, T^kv\}$ are linearly dependent. Hence $T^kv = \sum_{i=0}^{k-1} a_iT^iv$. Conclude that $\mathfrak{m}_v(x) = x^k - \sum_{i=0}^{k-1} a_ix^i$ is the required polynomial by division algorithm. □

Properties of order of v: *(Check the following:)*

(a) $\mathfrak{m}_v$ divides the minimal polynomial $\mathfrak{m}$ of T
(b) For a fixed $T \in L(V)$, consider the cyclic subspace generated by v $Z(v;T) = \{g(T)(v) : g \in F[x]\}$. Let $T_{Z(v;T)}$ be the restriction of T to $Z(v;T)$. Then order $\mathfrak{m}_v$ (of v) is the minimal polynomial of $T_{Z(v;T)}$ upto a constant factor.

5.2. **Companion matrix $\mathbb{A}_{p(x)}$ of a polynomial $p(x)$.**

Theorem 5.2. *Let $p(x) = x^k - a_{k-1}x^{k-1} - \cdots - a_0$ be an irreducible (prime) monic polynomial $\in F[x]$.*

(i) *Suppose $m_v(x) = p(x)$ is the order of v. Then the matrix of $T_{Z(v;T)}$ with respect to the basis $\{v, Tv, \cdots T^{k-1}v\}$ of $Z(v;T)$ is*

$$\mathbb{A}_{p(x)} := \begin{pmatrix} 0 & & & \cdots & & a_0 \\ 1 & 0 & & \cdots & & a_1 \\ 0 & 1 & 0 & \cdots & & \\ \vdots & & & \cdots & & \vdots \\ & & & & 0 & \\ 0 & & \cdots & 0 & 1 & a_{k-1} \end{pmatrix}.$$

Proof. By the definition of $Z(v;T)$ and the proof of Lemma 5.1, we have a basis $\{v, Tv, \cdots T^{k-1}v\}$ of $Z(v;T)$. Since $T(T^i(v)) = T^{i+1}v$ for all $i \leq k-2$ and

$$T(T^{k-1}v) = a_0v + a_1Tv + \cdots a_{k-1}T^{k-1}v,$$

clearly the matrix is as described. □

(ii) *Suppose $m_v(x) = p^r(x)$ for some integer $r > 1$. Then the matrix of $T_{Z(v;T)}$ with respect to the basis*

$$\{p(T)^{r-1}v, Tp(T)^{r-1}v, \cdots T^{k-1}p(T)^{r-1}v,$$
$$p(T)^{r-2}v, Tp(T)^{r-2}v, \cdots T^{k-2}p(T)^{r-1}v,$$
$$\cdots, v, Tv, \cdots T^{k-1}v\}$$

of $Z(v;T)$ is

$$\mathbb{A}_{\mathfrak{m}(x)} := \begin{matrix} \uparrow \\ r \textit{ blocks} \\ \downarrow \end{matrix} \begin{pmatrix} \tilde{\mathbb{A}} & \mathbb{B} & 0 & \cdots & 0 \\ 0 & \tilde{\mathbb{A}} & \mathbb{B} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & \cdots & \tilde{\mathbb{A}} \end{pmatrix},$$

where $\tilde{\mathbb{A}} = \mathbb{A}_{p(x)}$ and $B = (b_{ij})_{k\times k}$
with $b_{1k} = 1$ and $b_{ij} = 0$ for all other i, j

5.3. Rational and Jordan Canonical form.

Theorem 5.3. ***Elemntary divisor theorem - Cyclic decomposition theorem:***
Let $T \in L(V)$ where V is a finite dimensional vector space over an arbitrary field and $V \neq 0$. Then there exist non-zero vectors $\{v_1, v_2, \cdots v_k\}$ in V whose orders are powers of prime polynomials in $F[x]$, $\{p_1(x)^{r_1}, \cdots, p_k(x)^{r_k}\}$ and are such that V is the direct sum of the cyclic subspaces $Z(v_i;T)$. This is unique upto a rearrangement.

The outline of the proof was discussed in the class. We concentrate on the proof when the field is the complex numbers: For the proof, refer to [4] (Chapter XI: Section 6)
Recall the example 4.1: The minimal polynomial is $\mathfrak{m}(x) = (x-1)^2(x-4)$ and check that $N_{T-4I} = Z((1,3,9);T)$ and $N_{T-I} = Z((0,1,0);T)$. Note that $Z((1,0,0);T)$ is a subspace of $Z((0,1,0);T)$. Here rational form is same as the matrix obtained by the primary decomposition

Thought! -What is the rational canonical form (Jordan form if the Field is algebraically closed) of a nilpotent matrix?

5.4. Some insights on S-N decomposition.

For this subsection, let us study vector spaces on the field of complex numbers.
A *semisimple operator* T is defined to be a linear operator such that every T-invaraiant subspace has a complimentary T-*invariant* subspace, that is, if W is a T-invariant subspace of V, then there exists $\tilde{W}$ such that $V = W \bigoplus \tilde{W}$ and $\tilde{W}$ is T-invariant.

Lemma 5.4. *Let $T \in L(v)$ and $V = V_1 \bigoplus \cdots \bigoplus V_k$ be the primary decomposition. If W is a T-invariant subspace, then $W = W \cap V_1 \bigoplus \cdots \bigoplus W \cap V_k$. (Refer* [2] *section 7.5 for the proof)*

Lemma 5.5. *Let $T \in L(V)$. If $\mathfrak{m}(x) = x - \lambda$, then T is semisimple.*

Theorem 5.6. *Let V be a finite dimensional vector space over the field of complex numbers. $T \in L(V)$ is semi-simple if and only if T is diagonal.*

6. Inner Product Spaces and Spectral theorem

- Inner product space, Gram-Schmidt
—— *we see why the geometric dimension is equal to the algebraic dimension of the space we have studied so far, via orthogonality*
- linear functionals and adjoints
—— *we see the existence of adjoint operators*
- unitary operators, normal operators, self-adjoint operators, spectral theorem for self-adjoint operators.

NOTES: (Refer [2] Chapter 8 and Theorem 9 in Chapter 9)

7. Appendix

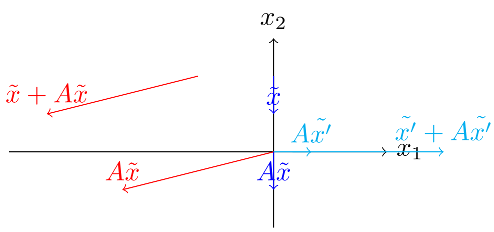
7.1. Example: Solution space of system of homogeneous linear equations. *M*otivation: (refer [3] page 3) Let us look at a simple example of differential equations. Consider the system of two differential equations in two unknown functions:

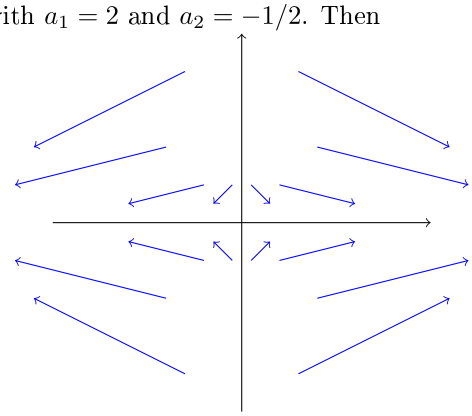
$$\begin{aligned} dx_1 &= a_1 x_1 dt, \\ dx_2 &= a_2 x_2 dt \end{aligned}$$

Many more complicated differential equations can be reduced to this simple system. You may study why in a later course! The solution is given by

$$\begin{aligned} x_1(t) &= C_1 exp(a_1 t),\ C_1 = \text{constant}, \\ x_2(t) &= C_2 exp(a_2 t),\ C_2 = \text{constant}. \end{aligned}$$

In fact the constants are determined by the initial conditions $x_1(t_0)$ and $x_2(t_0)$. Consider the map $x(t) = (x_1(t), x_2(t))$ from $\mathbb{R}$ to $\mathbb{R}^2$. Then the tangent vector $x'(t)$ to the curve $x(t)$ is given by the vector notation: $x' = Ax$, where $Ax = (a_1 x_1, a_2 x_2)$. Suppose we consider A the diagonal matrix with $a_1 = 2$ and $a_2 = -1/2$. Then





–as a vector field –
obtained from x to Ax

The solution curve of the differential equation can also be plotted and seen easily as 'a curved vector field'.

This picturization clearly describes why we look at the following system of equation given by $Ax = y$ or $Ax = 0$ where $A = (a_{ij})$ is an $m \times n$ matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

Homogeneous Linear equations:
We call the following the system of m linear equations in n unknowns: (we call it

homogeneous if $y_i = 0$, $\forall i$.

$$
(1) \qquad
\begin{array}{rcl}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &=& y_1 \\
A_{12}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &=& y_2 \\
\cdots\cdots\cdots && \cdots \\
\cdots\cdots\cdots && \cdots \\
A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &=& y_m
\end{array}
$$

Recall: If A is a $n \times n$ matrix,
the homogeneous system $AX = 0$ has only trivial solution $X = 0$
if and only if
the system of equations $AX = Y$ has a solution for each $n \times 1$ matrix Y
if and only if
A is invertible.

Solution space of a system of homogeneous linear equations:
Recall:

- Using the row-reduced echelon matrix ($RX = PY$ where the row reduced echelon matrix R equivalent to A is given by $R = PA$ where P is $m \times m$ invertible matrix), recall how did you conclude in your basic linear algebra class that:
  —— If A is an $m \times n$ matrix with $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.
  —— Suppose A is an $n \times n$ matrix. A is row-equivalent to $n \times n$ identity matrix if and only if the system of equations $AX = 0$ has only the trivial solution.
- recall for a non-homogeneous system of linear equations $AX = Y$ via augmented matrix $m \times (n+1)$ matrix $\tilde{A}$ (where $\tilde{A}_{ij} = A_{ij}$ if $j \leq n$ and $\tilde{A}_{i(n+1)} = A_i$)
- $A \in F^{n \times n}$ is invertible $<\Longrightarrow$ the homogeneous system $AX = 0$ has only trivial solution $<\Longrightarrow$ the system $AX = Y$ has a solution X for each $Y \in F^{n \times 1}$.
- Recall Cramer's rule!

Fix A, an $m \times n$ matrix over the field F. The solution space of $AX = 0$ is the set $\mathbb{S}$ of all $n \times 1$ matrices X over F that satisfies $AX = 0$.

Properties of Solution space:

(1) $\mathbb{S}$ is a vector space over F.
—It follows from the observation that $A_1(cA_2 + A_3) = c(A_1A_2) + A_1A_3$ for any $m \times n$ matrix A_i and $c \in F$.
(2) If R is row-reduced echelon matrix of A, then $S = S_R$ Solution space for the system $RX = 0$. (Important to know how to obtain basis of S_R from the basis of S) - Example 15, Section 2.3 [2].
(3) Homogeneous system of linear functions give the motivation to study linear functionals:
If we have the system $AX = 0$ where $A \in F^{m \times n}$, then consider $T_i(x_1, \cdots, x_n) = A_{i1}x_1 + \cdots + A_{in}x_n$. The solution space is given by $\{x = (x_1, \cdots, x_n) : T_i(x) = 0,\ i = 1, \cdots m\}$, that is subspace annihilated by $T_1, \cdots T_m$. Thus

row-reduction helps to give a systematic method of finding the annihilator of the subspace spanned by a given finite vectors in F^n.

7.1.1. *Solution space - differential equations.* (See Example 14 in section 6.8 - [2]):

Fix $F = \mathbb{C}$. Let n be a positive integer and W be the space of all n times continuously differentiable functions f on the real line which satisfy the differential equation

$$\frac{d^n f}{dx^n} + a_{n-1}\frac{d^{n-1} f}{dx^{n-1}} + \cdots + a_1 \frac{df}{dx} + a_0 f = 0$$

where $a_0, \cdots, a_{n-1}$ are some fixed constants. In fact, W is the null space of the operator $p(D)$ where $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is the polynomial and D denotes the differential operator. Also W is a $D-$ invariant subspace of the vector space V of all continuous functions.

Application of primary decomposition theorem 4.1: If $p(x) = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$, then by primary decomposition theorem $V = W_1 \bigoplus \cdots \bigoplus W_k$ where W_i is the null space of $(D - c_i I)^{r_i}$ and if f is the solution of the above differential equation, then $f = f_1 + \cdots + f_k$ where f_i's are the solutions of $(D - c_i I)^{r_i} f = 0$. Prove by induction that $f_i(x) = e^{c_i x}(b_0^{(i)} + b_1^{(i)} x + \cdots b_{r_i - 1} x^{r_i - 1})$ for some $b_j^{(i)} \in F$.

7.2. **Example: Space of all polynomials of any degree over the field F.**
Consider F to be either $\mathbb{R}$ or $\mathbb{C}$.
$F[x]$: the space of all polynomials from the field F into itself.
$F^m[x]$: the space of all polynomials of degree less than or equal to m from the field F into itself.
Check/Prove the following:

	$F[x]$	$F^m[x]$
Vector space: + pointwise addition	Yes	Yes. subspace of $F[x]$
Standard Basis $\mathcal{B}_0$	$\{1, x, x^2, \cdots\}$	$\{1, x, x^2, \cdots, x^m\}$
Dimension	Infinite	Finite $(= m+1)$
Identity: $I(x) = 1$	$I \in F[x]$	$I \in F^m[x]$
Differential operator, D $D(p)(x) = \sum_{j=0}^{k} jc_jx^{j-1}$ $\forall p(x) = \sum_{j=0}^{k} c_jx^j$	$D \in L(F[x])$	$D \in L(F^m[x])$ $(deg(p) = k$ must be $\leq m)$
N_D, Null space of D	Constant polynomials	Constant polynomials
$[D]_{\mathcal{B}_0}$	does not exist	see (2)
Integral operator, $\mathcal{I}$ $\mathcal{I}(p)(x) = \sum_{j=0}^{k} \frac{c_j}{j+1}x^{j+1}$ $\forall p(x) = \sum_{j=0}^{k} c_jx^j$	$\mathcal{I} \in L(F[x])$ $D\mathcal{I} = I$ $\mathcal{I}D \neq I$	$\mathcal{I} \notin L(F^m[x])$
$\mathcal{I}$ is right inverse of D	Yes	No
multiplication by x $M(p)(x) = xp(x)$	$M \in L(F[x])$	$M \notin L(F^m[x])$
Evaluation at x_0, L_{x_0} $(L_{x_0}(p) = p(x_0).)$	$L_{x_0} \in (F[x])^*$	$L_{x_0} \in (F^m[x])^*$
$L_a^b(p) = \int_a^b p(x)dx$ for fixed $a, b \in F$	$L \in (F[x])^*$	$L \in (F^m[x])^*$
$f_j(p) = c_j, \ \forall j$ where $p(x) = \sum_{j=0}^{k} c_jx^j$	$f_j \in (F[x])^*$	$f_j \in (F^m[x])^*$ deg $(p) \leq m$
	basis for $(F[x])*$?	$\{f_j\}_0^m$ is a basis for $(F^m[x])^*$
Express L_{x_0}, L_a^b in terms of the basis		

(2)

$$[T]_{\mathcal{B}_0} = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 2 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & m \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}$$

Let us denote the standard basis element $e_i(x) = x^{i-1}$ and $y \in F$. Define $\mathcal{B} = \{g_i\}$ where $g_i(x) = (x+t)^{i-1}$.

$$\begin{aligned} g_1 &= e_1 \\ g_2 &= te_1 + e_2 \\ g_3 &= t^2e_1 + 2te_2 + e_3 \\ & ... \\ g_m &= t^{m-1}e_1 + (m-1)t^{m-2}e_2 + (m-1)C_2t^{m-3}e_3 + \cdots + (m-1)te_{m-1} + e_m \end{aligned}$$

Hence

$$\begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 2 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & m \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} = [D]_{\mathcal{B}} = P[D]_{\mathcal{B},}P^{-1}, \text{where } P = \begin{bmatrix} 1 & t & t^2 & ... & t^{m-1} \\ 0 & 1 & 2t & ... & (m-1)t^{m-2} \\ 0 & 0 & 1 & ... & (m-1)C_2t^{m-3} \\ . & . & . & ... & .. \\ 0 & 0 & 0 & ... & (m-1)t \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

Other points to remember:

(1) There is exactly one polynomial function p over $\mathbb{R}$ which has degree at most m and satisfies $p(t_j) = c_j,\ j = 1, ...m+1$. Refer to the example 22 in the section 3.5 from [2] that we discussed during the class.

(2) We have discussed transpose of differential operator D (It acts on Linear functionals not on the vector space). See 2nd question in the quiz 1 (**??**).

(3) Check that the null space of $D - 0I$ is of dimension is 1 but the multiplicity of the polynomial $x - 0$ is 5 in the characteristic polynomial of D. Hence D is not diagonalizable.

7.3. **Quotient spaces.** Refer section 26 in [1].

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